

# The Optimal Dartboard?

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# Challenge

Find the best arrangement of numbers 1,2,...,20 on a dartboard.

## *Possibilities*

- There are  $20! \approx 10^{18}$  distinct arrangements.
- There are  $\frac{20!}{2 \times 20} \approx 10^{17}$  distinct cyclic groups.

## *Constraints*

- Penalise mistakes by over-ambitious players (Gamlin).
- Alternate odd and even numbers (Eastaway & Haigh).
- Impose rotational quasi-symmetry.

# Notation

Define cyclic group  $\{x_1, x_2, \dots, x_{20}\}$  and  $x_0 = x_{20}$  for convenience.



$\{20, 1, 18, 4, 13, 6, 10, 15, 2, 17, 3, 19, 7, 16, 8, 11, 14, 9, 12, 5\}$

# Differences

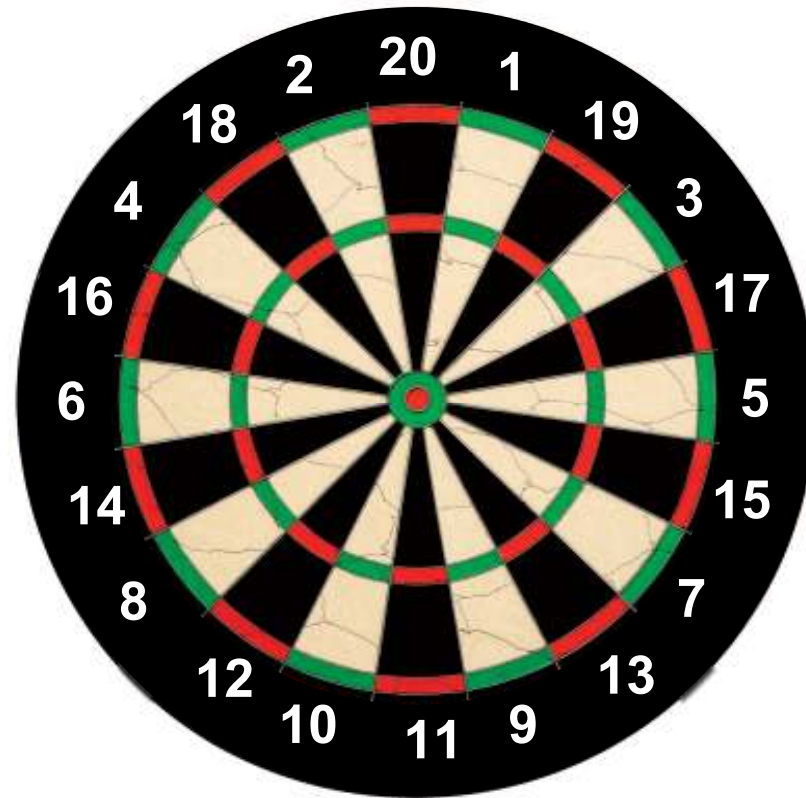
Maximise the  $p$ -norm  $\|\mathbf{d}\|_p = \left( \sum_{i=1}^{20} |d_i|^p \right)^{1/p}$  where  $d_i = x_i - x_{i-1}$ .

- $(10!)^2 / 40 \approx 10^{11}$  cyclic groups achieve  $\max \|\mathbf{d}\|_1 = 200$ .
- Unique cyclic group achieves  $\max \|\mathbf{d}\|_2 = \sqrt{2,642}$ .

*None of these solutions satisfy the parity constraint.*

*Differences do not penalize mistakes by over-ambitious players.*

# Singmaster's Dartboard



{20,1,19,3,17,5,15,7,13,9,11,10,12,8,14,6,16,4,18,2}

# Central Sums

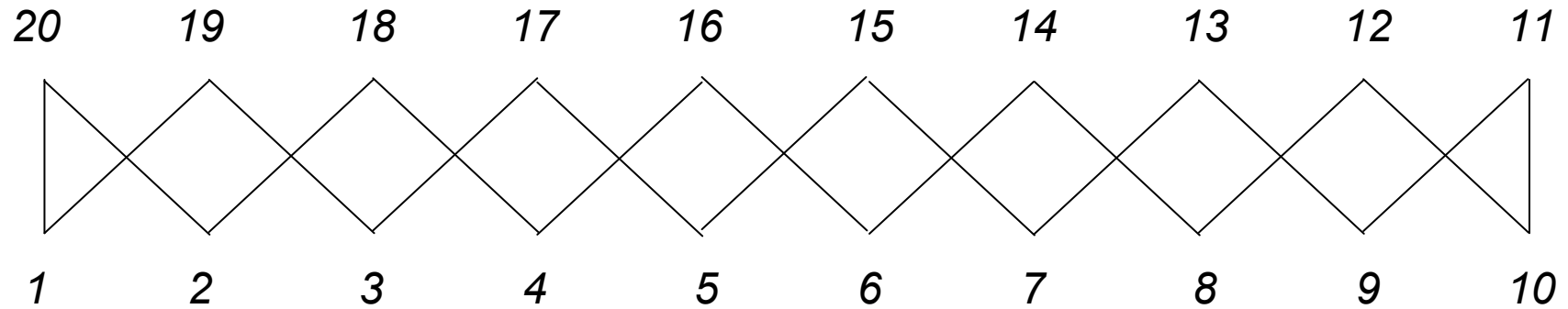
Minimise the  $p$ -norm  $\|\mathbf{s}\|_p = \left( \sum_{i=1}^{20} |s_i|^p \right)^{1/p}$  where  $s_i = x_i + x_{i-1} - 21$ .

- Several cyclic groups achieve likely minimum  $\|\mathbf{s}\|_1 = 18$ .
- Unique cyclic group achieves  $\min\|\mathbf{s}\|_2 = 18$ .

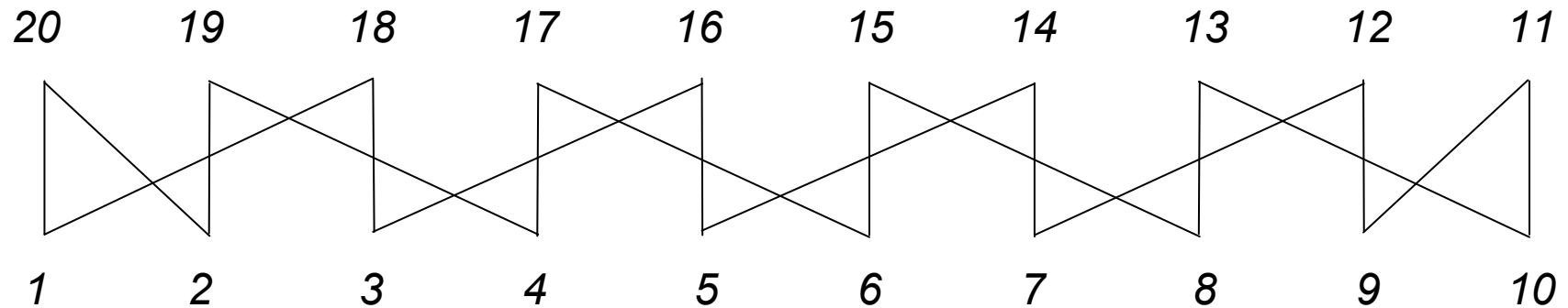
*Likely none of these solutions satisfy the parity constraint.*

*Aggregate measures do not impose rotational quasi-symmetry.*

# Examples with $\|s\|_1=18$



$$(2 \times 0) + (18 \times 1) = 18$$



$$(10 \times 0) + (2 \times 1) + (8 \times 2) = 18$$

# Better Measure

Minimise the maximum norm

$$\|\mathbf{s}\|_{\infty} = \lim_{p \rightarrow \infty} \left( \sum_{i=1}^{20} |s_i|^p \right)^{1/p} = \max_i |s_i|$$

where

$$s_i = x_i + x_{i-1} - 21$$

subject to the parity constraint

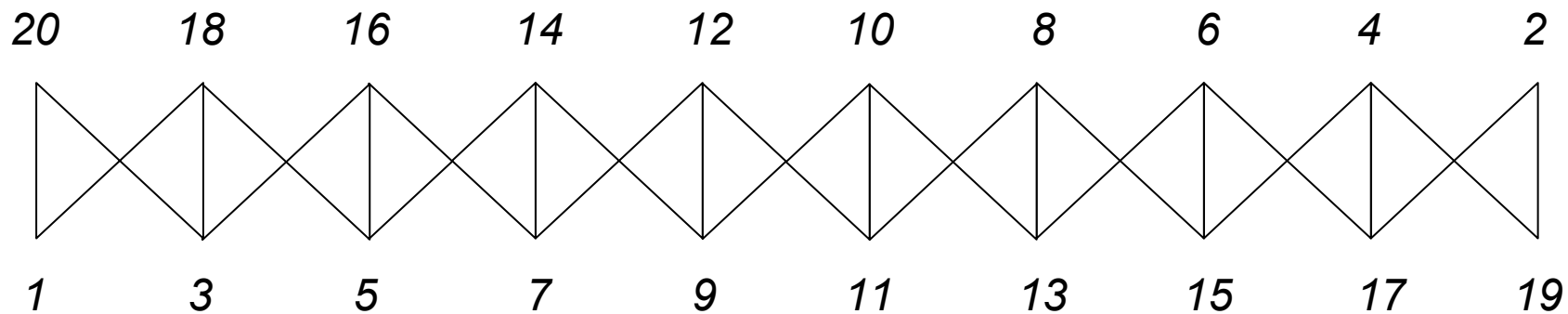
$$\frac{s_i}{2} \in \mathbb{Z}.$$

## Theorem

Unique parity-constrained cyclic group achieves  $\min \|\mathbf{s}\|_{\infty} = 2$ .

{20, 1, 18, 5, 14, 9, 10, 13, 6, 17, 2, 19, 4, 15, 8, 11, 12, 7, 16, 3}

## Proof



All pairs of even and odd numbers that sum to 19, 21 or 23.

Remove 8 edges to ensure that each vertex has degree 2.

Must remove all internal edges to ensure a cyclic group.



# Summary

	maximise		minimise		
	$\ \mathbf{d}\ _1$	$\ \mathbf{d}\ _2$	$\ \mathbf{s}\ _1$	$\ \mathbf{s}\ _2$	$\ \mathbf{s}\ _\infty$
Gamlin's dartboard	198	2,478	52	182	5
Singmaster's dartboard	200	2,642	18	18	1
<b>Percy's dartboard</b>	<b>198</b>	<b>2,588</b>	<b>36</b>	<b>72</b>	<b>2</b>

# References

- **Selkirk K. (1976)**  
Re-designing the dartboard, *Math. Gazette*, **60**, 171-178.
- **Singmaster D. (1980)**  
Arranging a dartboard, *IMA Bulletin*, **16**, 93-97.
- **Eiselt H.A. & Laporte G. (1991)**  
A combinatorial optimization problem arising in dartboard design, *J. Oper. Res. Soc.*, **42**, 113-118.
- **Everson P.J. & Bassom A.P. (1994)**  
Optimal arrangements for a dartboard, *Math. Spectrum*, **27**, 32-34.
- **Cohen G.L. & Tonkes E. (2001)**  
Dartboard arrangements, *Electron. J. Combin.*, **8**, R4.
- **Curtis S.A. (2004)**  
Darts and hoopla board design, *Info. Proc. Letters*, **92**, 53-56.
- **Liao Y.J., Shieh M.Z. & Tsai S.C. (2007)**  
Arranging numbers on circles to reach maximum total variations, *Electron. J. Combin.*, **14**, R47.
- **Eastaway R. & Haigh J. (2011)**  
*The Hidden Mathematics of Sport*, London: Portico.