On fair partitions of {1,2,...,n}

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(joint work with Phil Harvey)

At the last MathsJam, Phil posed the problem of partitioning {1,2,...,16} into two subsets of equal size so that each subset has the same sum, same sum of squares and same sum of cubes.

He showed that {1,4,6,7,10,11,13,16} {2,3,5,8,9,12,14,15} is a solution.

More generally, we say that a partition $\{1,2,...,n\} = A \cup B$ is *t*-fair if

$$\sum_{a \in A} a^i = \sum_{b \in B} b^i$$

for $i = 0,1,...,t$.

For example, $\{1,...,12\} = \{1,3,7,8,9,11\} \cup \{2,4,5,6,10,12\}$ is 2-fair because

$$1^0 + 3^0 + 7^0 + 8^0 + 9^0 + 11^0 = 2^0 + 4^0 + 5^0 + 6^0 + 10^0 + 12^0$$
$$1^1 + 3^1 + 7^1 + 8^1 + 9^1 + 11^1 = 2^1 + 4^1 + 5^1 + 6^1 + 10^1 + 12^1$$
$$1^2 + 3^2 + 7^2 + 8^2 + 9^2 + 11^2 = 2^2 + 4^2 + 5^2 + 6^2 + 10^2 + 12^2$$

There are two nice tricks for constructing new t-fair partitions from old.

TRICK 1. If $\{1,\ldots,m\} = A \cup B$ is t-fair and $\{1,\ldots,n\} = C \cup D$ is t-fair, then $\{1,\ldots,m+n\} = (A \cup (C+m)) \cup (B \cup (D+m))$ is t-fair.
(Proof is an easy exercise, using binomial theorem.)

Example

$\{1,\ldots,8\} = \{1,4,6,7\} \cup \{2,3,5,8\}$ is 2-fair
$\{1,\ldots,12\} = \{1,3,7,8,9,11\} \cup \{2,4,5,6,10,12\}$ is 2-fair
→
$\{1,\ldots,20\} = \{1,4,6,7,9,11,15,16,17,19\} \cup \{2,3,5,8,10,12,13,14,18,20\}$ is 2-fair.

A particular case of Trick 1 is to combine a t-fair partition $A \cup B$ of $\{1,\ldots,n\}$ with itself to get a t-fair partition of $\{1,\ldots,2n\}$.

A more cunning move is to combine $A \cup B$ with $B \cup A$, as this gives a $(t+1)$-fair partition of $\{1,\ldots,2n\}$.

TRICK 2. If $\{1,\ldots,n\} = A \cup B$ is t-fair,
then $\{1,\ldots,2n\} = (A \cup (B+n)) \cup (B \cup (A+n))$ is $(t+1)$-fair.
(Again, proof is an easy exercise using binomial theorem.)

Example

$\{1\} \cup \{2\}$ is 0-fair

→

$\{1,4\} \cup \{2,3\}$ is 1-fair

→

$\{1,4,6,7\} \cup \{2,3,5,8\}$ is 2-fair

→

$\{1,4,6,7,10,11,13,16\} \cup \{2,3,5,8,9,12,14,15\}$ is 3-fair

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$\{1,4,6,7,10,11,13,16,18,19,21,24,25,28,30,31\}\cup\{2,3,5,8,9,12,14,15,17,20,22,23,26,27,29,32\}$ is 4-fair, etc.

Theorem

There exists a 3-fair partitition of $\{1,...,n\}$ if and only if n is a multiple of 8 and $n \geq 16$.

Outline proof

For "if" part, use Trick 2 to construct base partitions for $n = 16$ and $n = 24$. Then use Trick 1 to combine these to get partitions for all larger multiples of 8.
For "only if" part, use parity arguments.

The problem of finding two distinct sets of integers having the same power sums for many powers has a long history. It is known as the Prouhet-Tarry-Escott problem, after Eugene Prouhet, who studied the problem in the 1850's, and Tarry and Escott, who studied it in the 1910's.

There are also links with the "ubiquitous" Prouhet-Thue-Morse sequence (1851) (1906) (1921)

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which occurs in combinatorics, differential geometry, number theory, group theory, mathematical physics, etc.