Hyperbolic Knots

MathsJam IV

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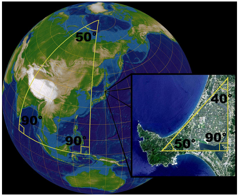
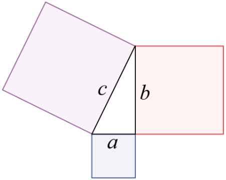
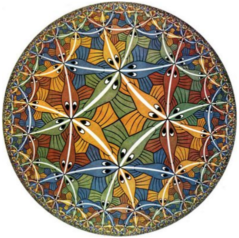
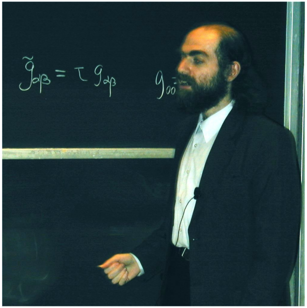
November 2013

Two-dimensional geometry

Spherical/Elliptic	Euclidean	Hyperbolic
		
$\mathbb{S}^2$	$\mathbb{E}^2$	$\mathbb{H}^2$
positive curvature	flat	negative curvature
$\triangle > \pi$	$\triangle = \pi$	$\triangle < \pi$

Three-dimensional geometry

William Thurston (1946–2012)



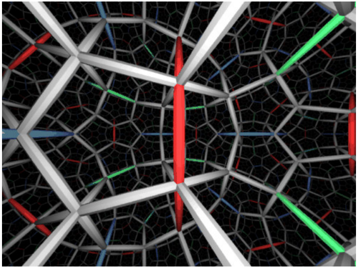
Grigori Perelman

Eight 3–dimensional geometries:

$$\mathbb{S}^3 \qquad \mathbb{E}^3 \qquad \mathbb{H}^3$$

$$\mathbb{S}^2\times\mathbb{R} \qquad \mathbb{H}^2\times\mathbb{R}$$

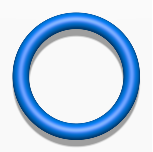
$$\widetilde{\mathrm{SL}}_2(\mathbb{R}) \qquad \mathrm{Nil} \qquad \mathrm{Sol}$$

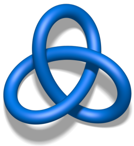
He talked of his dreams in a strangely poetic fashion; making me see with terrible vividness the damp Cyclopean city of slimy green stone – whose geometry, he oddly said, was *all wrong* – and hear with frightened expectancy the ceaseless, half-mental calling from underground: "Cthulhu fhtagn", "Cthulhu fhtagn."

– H P Lovecraft, *The Call of Cthulhu* (1926)

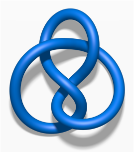
A knot is an embedded circle in 3–space.



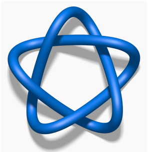
0_1



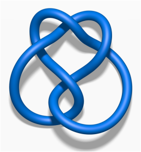
3_1



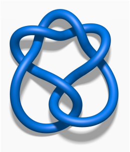
4_1



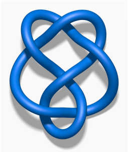
5_1



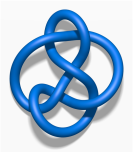
5_2



6_1



6_2



6_3

There are 1 701 936 prime knots with up to 16 crossings.


JIM HOSTE, MORWEN THISTLETHWAITE, AND JEFF WEEKS

The First 1,701,936 Knots

The history of knot tabulation is long established, having begun over 120 years ago. In many ways, the compilations of the first knot tables marked the beginning of the modern study of knots, and it is perhaps not surprising that as knot theory and topology grew, so did the knot tables. Over the last few years, we have extended the tables to include all prime knots with 16 or fewer crossings. This represents more than a 130-fold increase in the number of tabulated knots since the last burst of tabulation that took place in the early 1980s. With more than 1.7 million knots now in the tables, we hope that the census will serve as a rich source of examples and counterexamples and as a general testing ground for our collective intuition. To this end, we have written a UNIX-based computer program called *KnotScape* which allows easy access to the tables.

The account of our methodology is prefaced by a brief history of knot tabulation, concentrating mostly on events taking place within the last 30 years. The survey article [Thi1] contains further details on the work of the nineteenth-century tabulators, but, above all, the reader is encouraged to consult the original sources, in particular the excellent series of papers by Tait [Tai]. Kirkman's papers make fascinating reading, as they abound with original ideas and ornate language—his definition of the term "knot' is a single sentence of 101 words. Conway's landmark paper [Con] is also highly recommended.

An important feature of our project is that we have worked in two completely separate teams, producing two tabulations which were kept secret until after they were complete.[1] Although it would be foolhardy to claim with absolute certainty that our tables are correct, we must report the gratifying experience of finding that our lists of 1,701,936 knots were in complete agreement! Moreover, we did not use exactly the same methods, the primary difference being the use of hyperbolic geometry by Hoste and Weeks and the complete absence of hyperbolic invariants in Thistlethwaite's approach. Nevertheless, our overall programs are similar in spirit and differ little from the methods of most tabulators who precede us. As part of our tabulation, using Weeks's program *SnapPea* we were able to compute the *symmetry groups* of the knots; we have included a short introduction to this beautiful and intriguing topic.

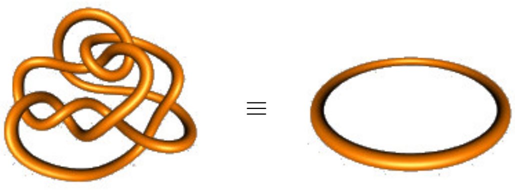
[1]At a recent conference, people who were aware of our project jokingly scolded us for conversing together.

 33

Crossings	Number	Crossings	Number
0	1	9	49
1	0	10	165
2	0	11	552
3	1	12	2 176
4	1	13	9 988
5	2	14	46 972
6	3	15	253 293
7	7	16	1 388 705
8	21		

J Hoste, **M Thistlethwaite**, **J Weeks**, *The First 1 701 936 Knots*, Math. Intelligencer 20 (1998) 33–48

It's not always easy to recognise which knot is which.

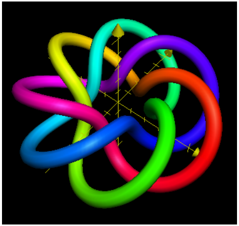
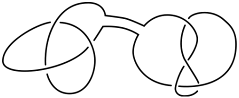
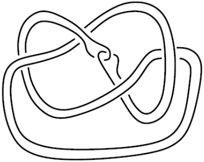


We need sophisticated techniques to compare and distinguish knots that might be the same.

- Colouring number
- Alexander polynomial Δ_K
- Conway polynomial ∇_K
- Jones polynomial V_k
- Kauffman bracket $\langle K \rangle$
- Genus
- Fibredness
- ...

Study the space *around* the knot instead.

Thurston discovered that the space around a knot ($S^3 \setminus K$) has a canonical geometry.
In "most" cases this geometry is $\mathbb{H}^3$.
The exceptions are torus knots and satellite knots

Of the first 1 701 936 knots, only 12 are torus and 20 are satellite.
The other 1 701 904 have complements with finite volume.
This hyperbolic volume $\mathrm{vol}(K) \in \mathbb{R}^+$ is a powerful invariant.