



Let  $\varepsilon = 1$

Paul Walter  
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If  $\varepsilon$  approaches zero...

$$f'(x) = \frac{f(x + \varepsilon) - f(x)}{\varepsilon}$$

$$\int_0^x f(t) dt = \sum_{n=1}^{x/\varepsilon} f(n\varepsilon) \times \varepsilon$$

...substituting  $\varepsilon = 1$ ...

$$f'(x) = \frac{f(x + \varepsilon) - f(x)}{\varepsilon}$$

$$f'(x) = f(x + 1) - f(x)$$

$$\int_0^x f(t) dt = \sum_{n=1}^{x/n} f(n\varepsilon) \times \varepsilon$$

$$\int_0^x f(t) dt = \sum_{n=1}^x f(n)$$



Let's practice:  $f(x) = x^2$

<b>x</b>	<b>1</b>	<b>2</b>	<b>3</b>	<b>4</b>	<b>5</b>	<b>6</b>	<b>7</b>	<b>8</b>	<b>9</b>
<b>f(x)</b>	<b>1</b>	<b>4</b>	<b>9</b>	<b>16</b>	<b>25</b>	<b>36</b>	<b>49</b>	<b>64</b>	<b>81</b>

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<b>x</b>	<b>1</b>	<b>2</b>	<b>3</b>	<b>4</b>	<b>5</b>	<b>6</b>	<b>7</b>	<b>8</b>	<b>9</b>
<b>f(x)</b>	<b>1</b>	<b>4</b>	<b>9</b>	<b>16</b>	<b>25</b>	<b>36</b>	<b>49</b>	<b>64</b>	<b>81</b>
<b>f'(x)</b>	<b>3</b>	<b>5</b>	<b>7</b>	<b>9</b>	<b>11</b>	<b>13</b>	<b>15</b>	<b>17</b>	<b>19</b>

Let's practice:  $f(x) = x^2$

<b>x</b>	<b>1</b>	<b>2</b>	<b>3</b>	<b>4</b>	<b>5</b>	<b>6</b>	<b>7</b>	<b>8</b>	<b>9</b>
<b>f(x)</b>	<b>1</b>	<b>4</b>	<b>9</b>	<b>16</b>	<b>25</b>	<b>36</b>	<b>49</b>	<b>64</b>	<b>81</b>
<b>f'(x)</b>	<b>3</b>	<b>5</b>	<b>7</b>	<b>9</b>	<b>11</b>	<b>13</b>	<b>15</b>	<b>17</b>	<b>19</b>
<b><math>\int f(x)dx</math></b>	<b>1</b>	<b>5</b>	<b>14</b>	<b>30</b>	<b>55</b>	<b>91</b>	<b>140</b>	<b>204</b>	<b>285</b>



<b>x</b>	<b>1</b>	<b>2</b>	<b>3</b>	<b>4</b>	<b>5</b>	<b>6</b>	<b>7</b>	<b>8</b>	<b>9</b>
<b>f(x)</b>	<b>1</b>	<b>4</b>	<b>9</b>	<b>16</b>	<b>25</b>	<b>36</b>	<b>49</b>	<b>64</b>	<b>81</b>
<b>f'(x)</b>	<b>3</b>	<b>5</b>	<b>7</b>	<b>9</b>	<b>11</b>	<b>13</b>	<b>15</b>	<b>17</b>	<b>19</b>
$\int_0^x f(x) dx$	<b>1</b>	<b>5</b>	<b>14</b>	<b>30</b>	<b>55</b>	<b>91</b>	<b>140</b>	<b>204</b>	<b>285</b>

$$f'(x) = 2x + 1 \quad (\text{close...})$$

$$\int f(x) dx = \frac{x^3}{3} + \frac{x^2}{2} + \frac{x}{6} + C$$

# Fundamental Theorem of Calculus

$$\left( \int_0^x f(x) dx \right)' = \left( \sum_{n=1}^x f(n) \right)' = \sum_{n=1}^{x+1} f(n) - \sum_{n=1}^x f(n) = f(x+1)$$

$$\int_0^x f'(t) dt = \int_0^x (f(t+1) - f(t)) dt = \sum_{n=1}^x f(n+1) - f(n) = f(x+1) - f(1)$$

# Let's take an example...

x	1	2	3	4	5	6	7	8	9	
g(x)	0	0	1	4	10	20	35	56	120	$\frac{(x-2)(x-1)x}{6}$
g'(x)	0	1	3	6	10	15	21	28	36	$\frac{(x-1)x}{2}$
g''	1	2	3	4	5	6	7	8	9	x

...and try to get it back...

x	1	2	3	4	5	6	7	8	9
	1	2	3	4	5	6	7	8	9
	1	3	6	10	15	21	28	36	45
	1	4	10	20	35	56	84	120	165

$x$

$\frac{x(x+1)}{2}$

$\frac{x(x+1)(x+2)}{6}$

**Let**

$$x^{\rightarrow n} = \frac{1}{n!} x(x+1)(x+2)\dots(x+n-1) = \binom{x+n-1}{n}$$

$$x^{\leftarrow n} = \frac{1}{n!} x(x-1)(x-2)\dots(x-n+1) = \binom{x}{n}$$

**We have**

$$\int x^{\rightarrow n} dx = x^{\rightarrow n+1} + C$$

$$\left( x^{\leftarrow n} \right)' = x^{\leftarrow n-1}$$



<b>x</b>	<b>1</b>	<b>2</b>	<b>3</b>	<b>4</b>	<b>5</b>	<b>6</b>	<b>7</b>	<b>8</b>	<b>9</b>
<b>f(x)</b>	1	32	145	418	953	1876	3337	5510	8593
<b>f'(x)</b>	31	113	273	535	923	1461	2173	3083	
<b>f''(x)</b>	82	160	262	388	538	712	910		
<b>f'''</b>	78	102	126	150	174	198			
<b>f''''</b>	24	24	24	24	24				

<b>x</b>		<b>1</b>	<b>2</b>	<b>3</b>	<b>4</b>	<b>5</b>	<b>6</b>	<b>7</b>	<b>8</b>	<b>9</b>
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<b>f'''</b>		78	102	126	150	174	198			
<b>f''''</b>		24	24	24	24	24				
<b>Integrate</b>		24	48	72	96	120				

x		1	2	3	4	5	6	7	8	9
f(x)		1	32	145	418	953	1876	3337	5510	8593
f'(x)		31	113	273	535	923	1461	2173	3083	
f''(x)		82	160	262	388	538	712	910		
f'''		78	102	126	150	174	198			
f''''		24	24	24	24	24				
Integrate		24	48	72	96	120				
	+78	102	126	150	174	198				

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f'(x)		31	113	273	535	923	1461	2173	3083	
f''(x)		82	160	262	388	538	712	910		
f'''		78	102	126	150	174	198			
f''''		24	24	24	24	24				
Integrate		24	48	72	96	120				
	+78	102	126	150	174	198				
Integrate		102	228	378	562	760				
	+160	262	388	538	712	910				

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f(x)		1	32	145	418	953	1876	3337	5510	8593
f'(x)		31	113	273	535	923	1461	2173	3083	
f''(x)		82	160	262	388	538	712	910		
f'''		78	102	126	150	174	198			
f''''		24	24	24	24	24				
Integrate		24	48	72	96	120				
	+78	102	126	150	174	198				
Integrate		102	228	378	562	760				
	+160	262	388	538	712	910				
Integrate		262	650	1188	1900	2810				
	+273	535	923	1461	2173	3083				

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f(x)		1	32	145	418	953	1876	3337	5510	8593
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f''(x)		82	160	262	388	538	712	910		
f'''		78	102	126	150	174	198			
f''''		24	24	24	24	24				
Integrate		24	48	72	96	120				
	+78	102	126	150	174	198				
Integrate		102	228	378	562	760				
	+160	262	388	538	712	910				
Integrate		262	650	1188	1900	2810				
	+273	535	923	1461	2173	3083				
Integrate		535	1458	2919	5092	8175				
	+418	953	1876	3337	5510	8593				

x		1	2	3	4	5	6	7	8	9
f(x)		1	32	145	418	953	1876	3337	5510	8593
f'(x)		31	113	273	535	923	1461	2173	3083	
f''(x)		82	160	262	388	538	712	910		
f'''		78	102	126	150	174	198			
f''''		24	24	24	24	24				
Integrate		24	48	72	96	120	$24x^{-1}$			
	<b>+78</b>	102	126	150	174	198	$24x^{-1} + 78$			
Integrate		102	228	378	562	760	$24x^{-2} + 78x^{-1}$			
	<b>+160</b>	262	388	538	712	910	$24x^{-2} + 78x^{-1} + 160$			
Integrate		262	650	1188	1900	2810	$24x^{-3} + 78x^{-2} + 160x^{-1}$			
	<b>+273</b>	535	923	1461	2173	3083	$24x^{-3} + 78x^{-2} + 160x^{-1} + 273$			
Integrate		535	1458	2919	5092	8175	$24x^{-4} + 78x^{-3} + 160x^{-2} + 273x^{-1}$			
	<b>+418</b>	953	1876	3337	5510	8593				

$$f(x) = 24(x - 4)^{-4} + 78(x - 4)^{-3} + 160(x - 4)^{-2} + 273(x - 4) + 418$$



# Quadratic Sequences

x		1	2	3	4	5	6	7	8	9
f(x)		3	5	13	27	47	73	105	143	187
f'(x)		2	8	14	20	26	32	38	44	
f''(x)	6	6	6	6	6	6	6	6		

$$f(x) = 6(x-2)^{-2} + 2(x-2) + 5 = 6 \frac{(x-2)(x-1)}{2} + 2x + 1 = 3x^2 - 7x + 7$$

# Dessert

Real calculus

$$(a^x)' = a^x \ln a$$

Integer calculus

$$(a^x)' = a^{x+1} - a^x = a^x(a-1)$$

# Dessert

## Real calculus

$$(a^x)' = a^x \ln a$$

$$(e^x)' = e^x$$

## Integer calculus

$$(a^x)' = a^{x+1} - a^x = a^x(a-1)$$

$$(2^x)' = 2^x$$

...so if  $\varepsilon = 1$  then  $e = 2!$

Real calculus

$$(a^x)' = a^x \ln a$$

$$(e^x)' = e^x$$

Integer calculus

$$(a^x)' = a^{x+1} - a^x = a^x(a-1)$$

$$(2^x)' = 2^x$$



**Thank you for your attention.**

**Paul Walter**  
**paulwalter@zoho.com**