

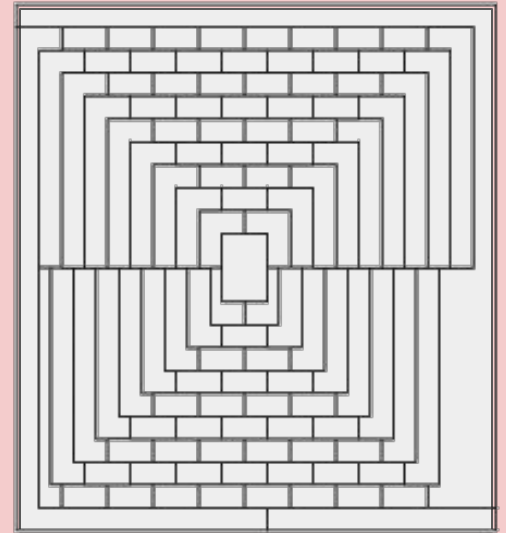
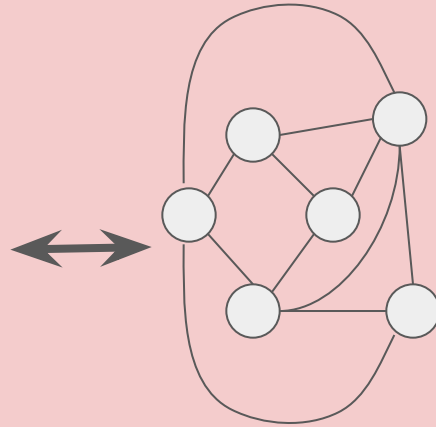
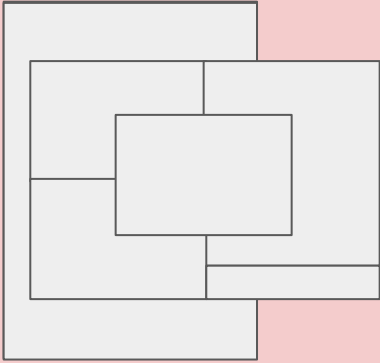
# The $4\frac{1}{2}$ -colour theorem

Alistair Bird

# The 4-colour theorem

Any map can be coloured with four colours so that each touching region doesn't share a colour.

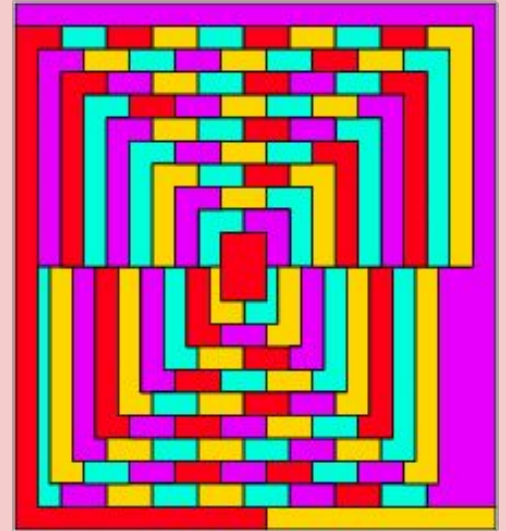
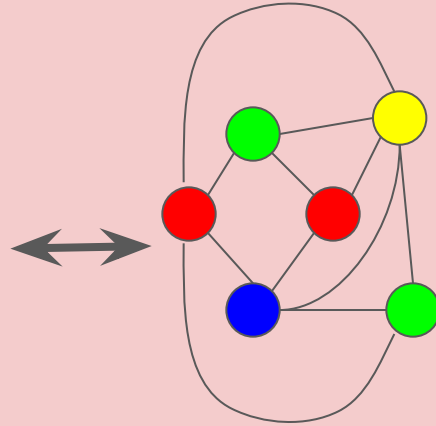
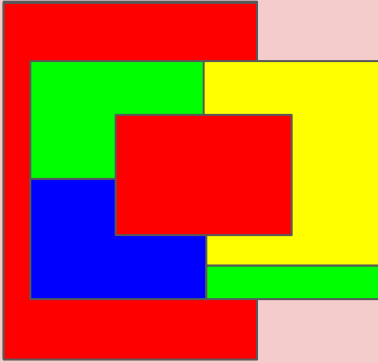
They are also *simple planar* graphs.



# The 4-colour theorem

Any map can be coloured with four colours so that each touching region doesn't share a colour.

They are *simple planar graphs*



# The 5-colour theorem

**Euler's formula:**

simple, planar graph has

$$v - e + f = 2$$

Every 'face' has at least 3 edges

Every edge has at most 2 faces,

$$\text{so } 3f \geq 2e.$$

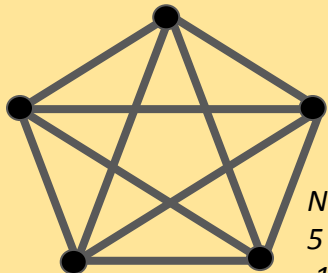
$$3v - 6 \geq e$$

$$(\text{for } v \geq 3)$$

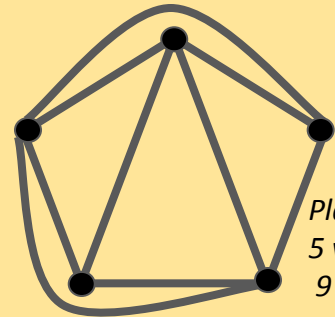
So  $K_5$  is non-planar because:

$$3 \times 5 \text{ vertices} - 6 = 9$$

$$\leq 10 \text{ edges}$$



*Non-planar:  
5 vertices,  
10 edges*



*Planar:  
5 vertices,  
9 edges*

# The 5-colour theorem

**Euler's formula:**

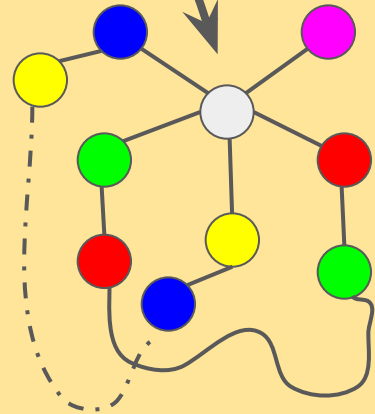
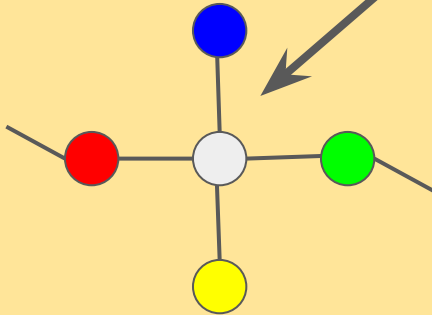
simple, planar graph has

$$v - e + f = 2$$

$$3v - 6 \geq e$$

(for  $v \geq 3$ )

There must be some vertex with five or fewer neighbours..



# Four colours: true



CTL #111



CTL #112



CTL #113

#421, 422, or  
423 at U



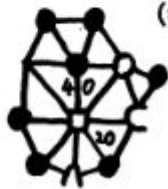
CTL #116

one of ##  
421, ..., 428  
at Q



CTL #117

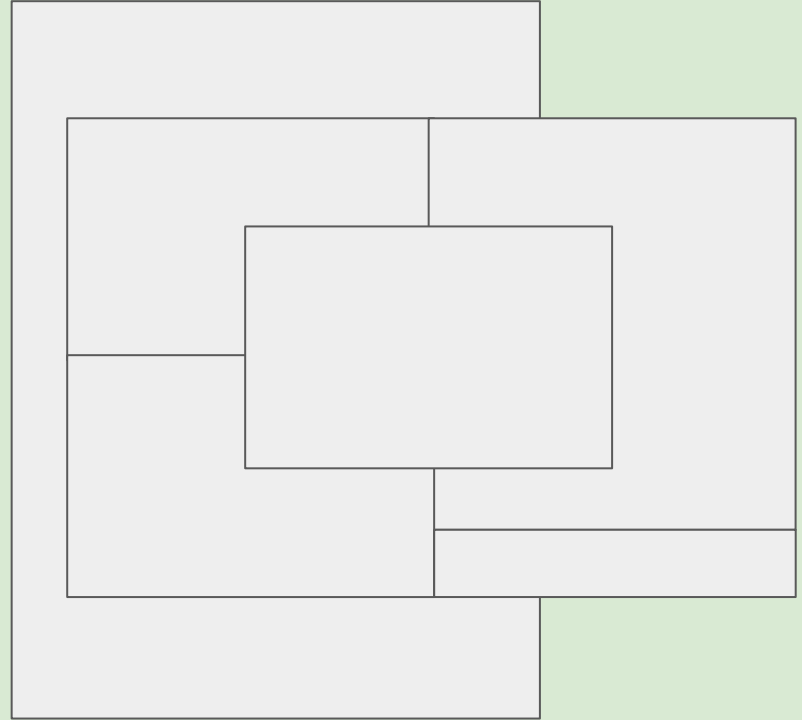
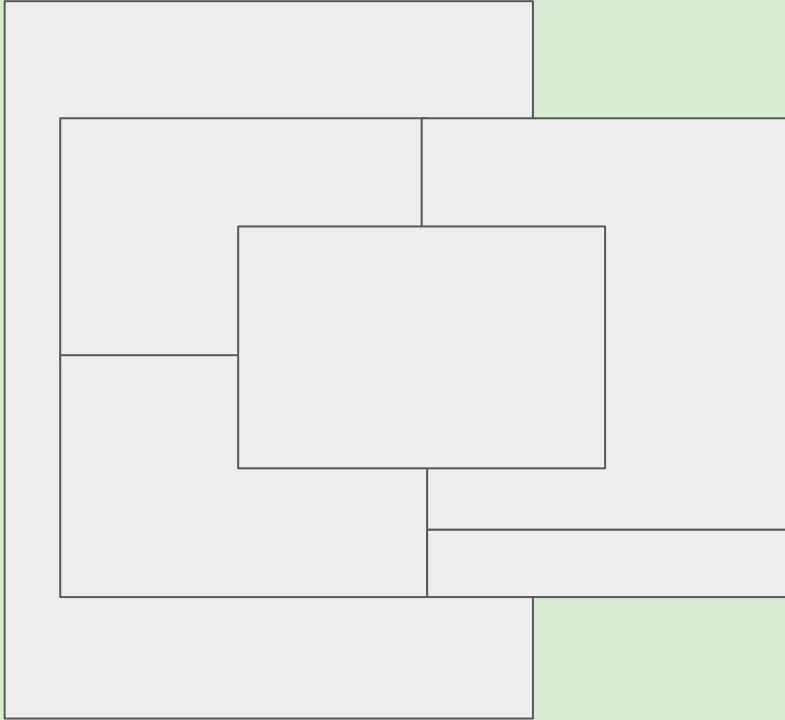
(#426)



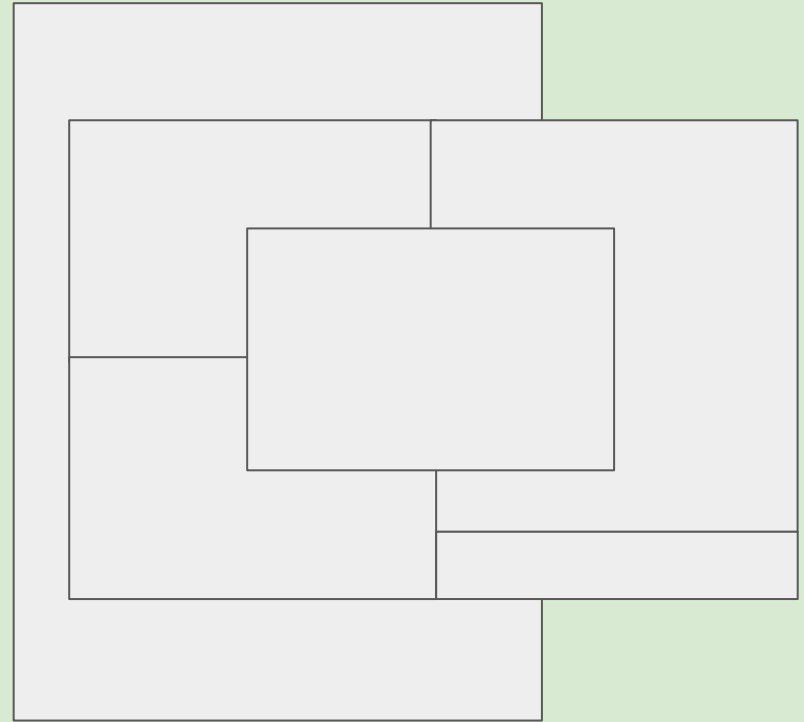
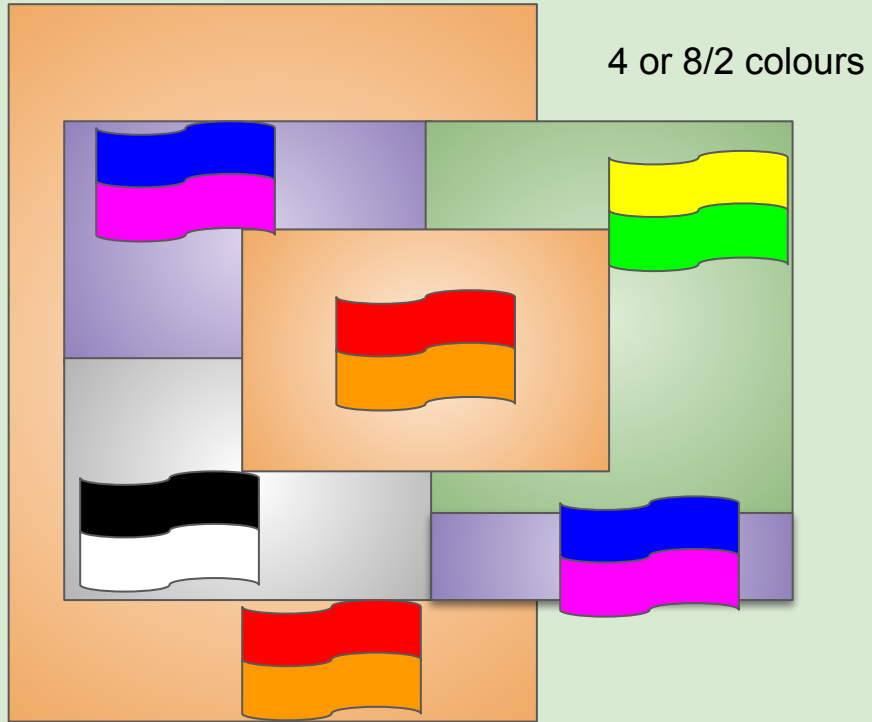
CTL #118

```
23 Section FourColorTheorem.
24
25 Variable Rmodel : Real.model.
26 Let R := Real.model_structure Rmodel.
27 Implicit Type m : map R.
28
29 Theorem four_color_finite m : finite_simple_map m -> colorable_with 4 m.
30 Proof.
31 intros fin_m.
32 pose proof (discretize.discretize_to_hypermap fin_m) as [G planarG colG].
33 exact (colG (combinatorial4ct.four_color_hypermap planarG)).
34 Qed.
35
36 Theorem four_color m : simple_map m -> colorable_with 4 m.
37 Proof. revert m; exact (finitize.compactness_extension four_color_finite). Qed.
38
39 End FourColorTheorem.
```

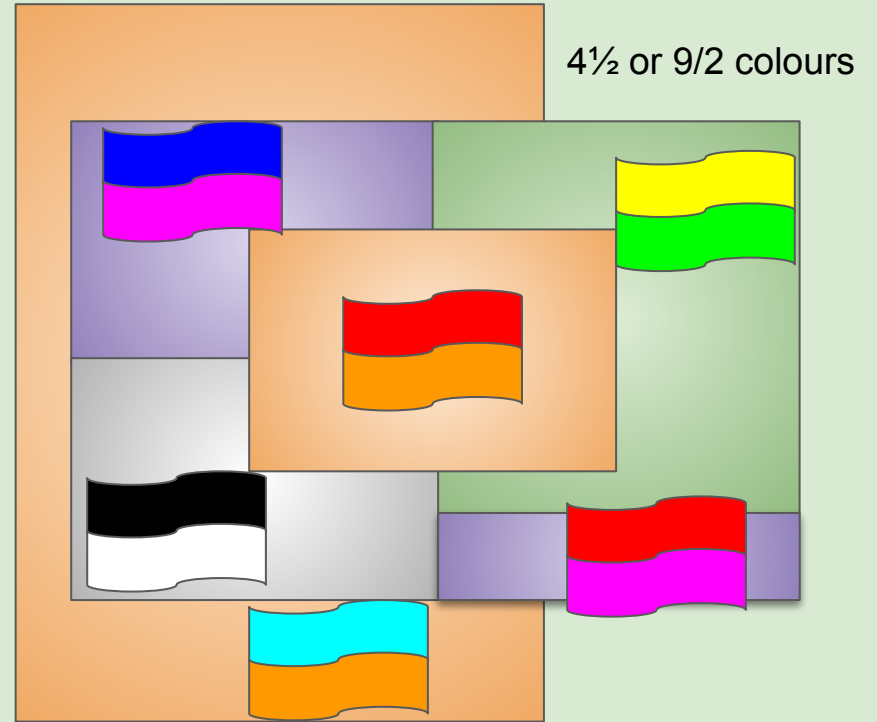
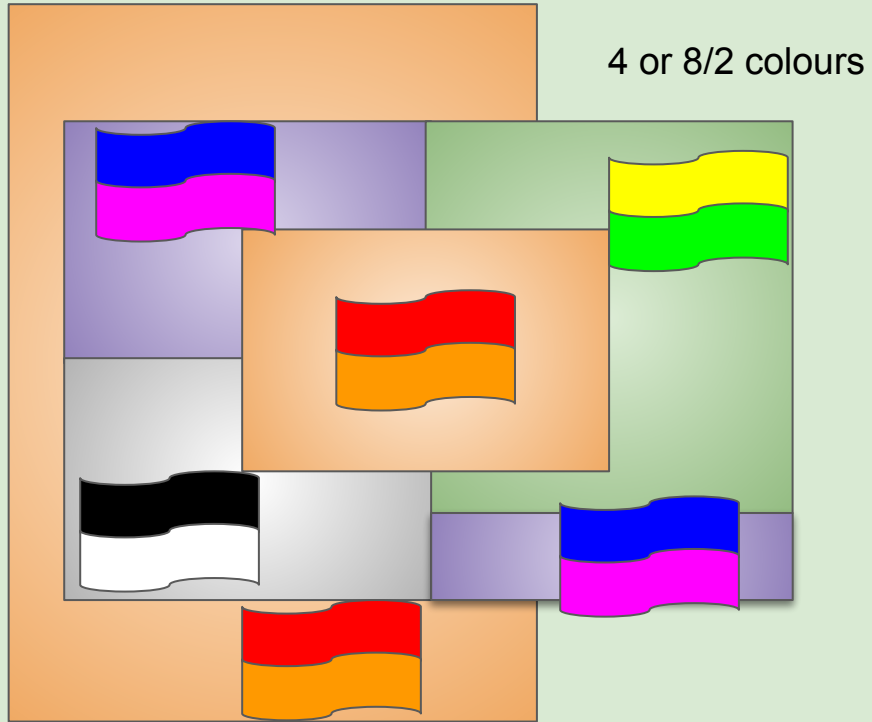
# The $4\frac{1}{2}$ -colour theorem



# The $4\frac{1}{2}$ -colour theorem



# The $4\frac{1}{2}$ -colour theorem



# Further Reading

Planar graphs are  $9/2$ -colorable  
and have big independent sets

Daniel W. Cranston

<http://www.people.vcu.edu/~dcranston/slides/planar-fractional-talk.pdf>

## PLANAR GRAPHS ARE $9/2$ -COLORABLE

DANIEL W. CRANSTON AND LANDON RABERN

ABSTRACT. We show that every planar graph  $G$  has a 2-fold 9-coloring. In particular, this implies that  $G$  has fractional chromatic number at most  $\frac{9}{2}$ . This is the first proof (independent of the 4 Color Theorem) that there exists a constant  $k < 5$  such that every planar  $G$  has fractional chromatic number at most  $k$ .

<https://arxiv.org/abs/1410.7233>

<https://www.ams.jhu.edu/ers/wp-content/uploads/sites/2/2015/12/fgt.pdf>

