$$\sqrt{2^{\frac{2}{3}}} = 2\sqrt{\frac{2}{3}}$$

Roman Mathematics
(Also Paris, Madrid, Lisbon, ???)

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Annual MathsJam Gathering 2019

Why might this matter to you?

If you are involved in:

- Entrance exams / placement tests
- Colleagues / students / parents from overseas
- Scicomm with an international audience
- Working overseas

Hope things in this talk will make you go “hmm”.

- If you want to investigate other examples of this take note of the messiness in this one

Objectives

- Awareness that unmixed countries exist
- Awareness that it’s not *that* clean and simple
- Self-consciousness in future
- I want info about other unmixed countries

In about 1990…
(the year may turn out to matter)

$$1\frac{1}{4}$$

I wrote this on a blackboard in Rome University maths department as my final answer to something

Audience reaction

$$1\frac{1}{4}$$

Your answer is wrong. And why have you written one quarter in such an eccentric way?

People said

- This notation does not exist
- You’ve made it up
- We don’t believe you. Bring us a book where this is used.

But someone in education dept...

These people are all too young. People would need to be about 30 years older than you to have done mixed numbers at school.

When/why/how did they vanish? Not known.

Something something fascism something?

Abolition? Natural death?

2018, MaddMaths!

$$\sqrt{2^{\frac{2}{3}}} = 2\sqrt{\frac{2}{3}}$$

Some reactions to first slide in Italy

- This image should be removed for the good of teaching.
- How do you know if it’s a sum or a product?
- Which books use this notation? I’ve never seen it before.

Some more reactions

- I have a degree in maths but I learned this notation in the UK.
- I had never seen this until I went to work for an international school.
- I had never seen this until I started tutoring people for US university admission exams. It’s the number one topic people ask about.
- It’s a special notation used in combinatorial game theory.

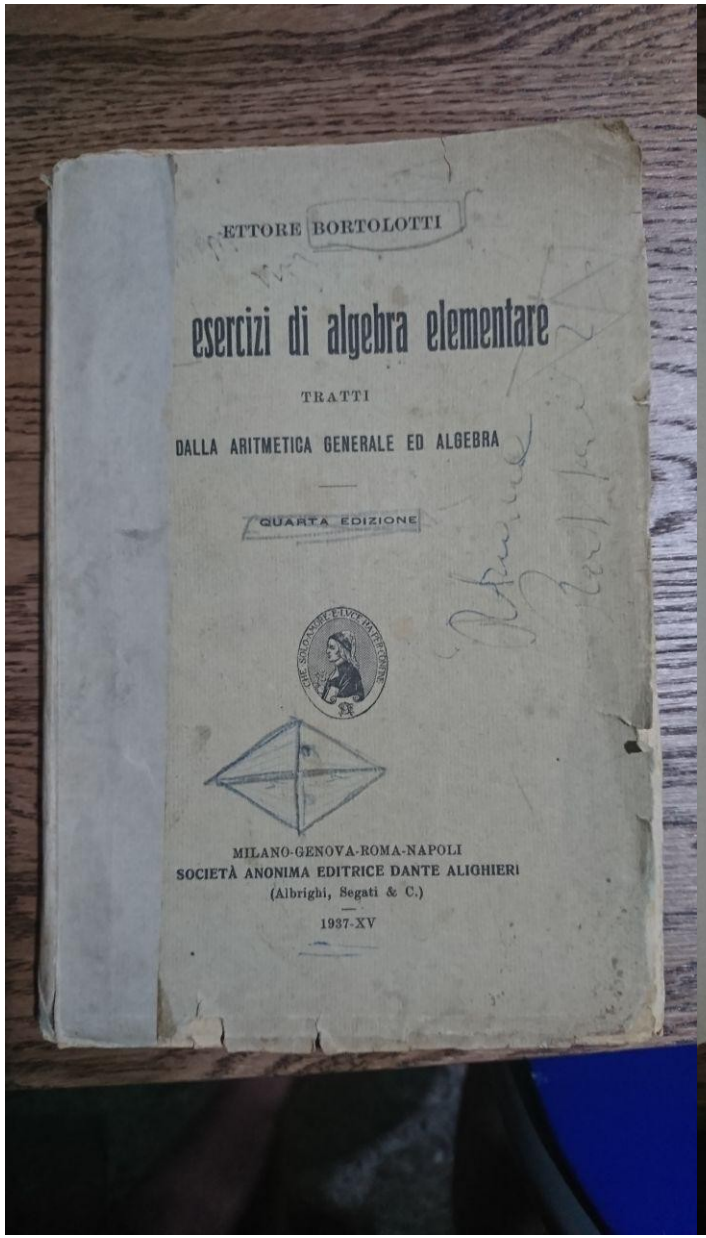
EVIL

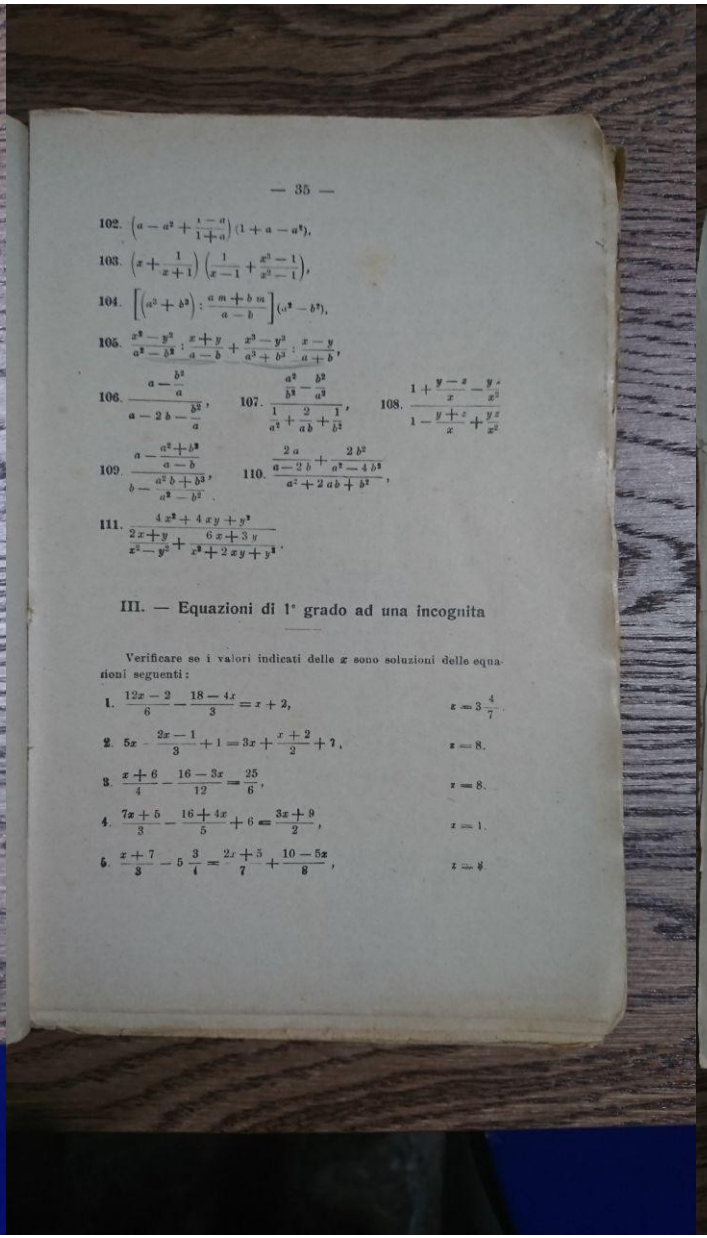
IMMORAL

Investigating...

- I started school in 1936 and we never did mixed numbers.
- “two and two thirds” 6.5 million Google hits
- “due e due terzi” 59000 hits, many *very* old
- I just thought 5 ¼ and 3 ½ disks were their names. I didn’t think they meant anything.
- You can say them but you can’t write them down
- You can’t even say them. It’s not good Italian.

Book from 1937 has them





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102. $\left(a - a^2 + \frac{1-a}{1+a}\right)(1 + a - a^2)$,

103. $\left(x + \frac{1}{x+1}\right)\left(\frac{1}{x-1} + \frac{x^2-1}{x^3-1}\right)$,

104. $\left[\left(a^2+b^2\right) : \frac{a\,m + b\,m}{a-b}\right](a^2-b^2)$,

105. $\frac{x^2-y^2}{a^2-b^2} : \frac{x+y}{a-b} + \frac{x^3-y^3}{a^3+b^3} : \frac{x-y}{a+b}$,

106. $\frac{a - \frac{b^2}{a}}{a - 2b - \frac{b^2}{a}}$, 107. $\frac{\frac{a^2}{b^2} - \frac{b^2}{a^2}}{\frac{1}{a^2} + \frac{2}{ab} + \frac{1}{b^2}}$, 108. $\frac{1 + \frac{y-x}{x} - \frac{y}{x^2}}{1 - \frac{y+x}{x} + \frac{y^2}{x^2}}$

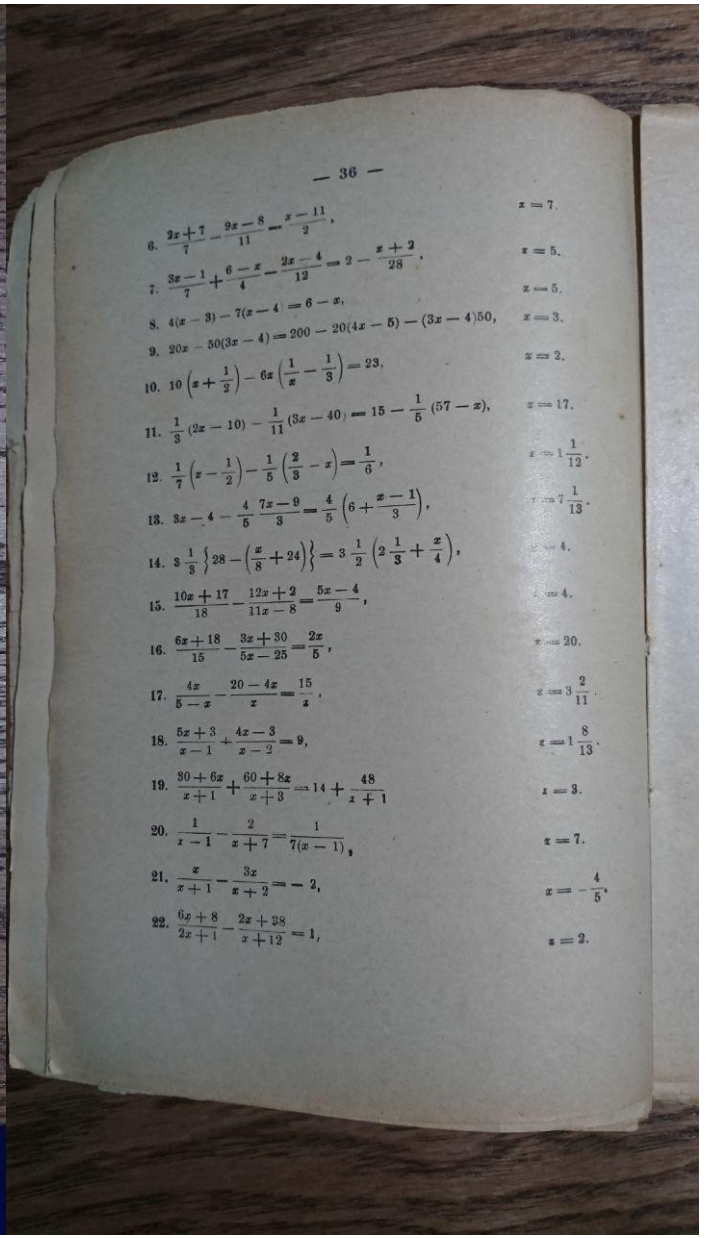
109. $\frac{a - \frac{a^2+b^2}{a-b}}{b - \frac{a^2b+b^3}{a^2-b^2}}$, 110. $\frac{\frac{2a}{a-2b} + \frac{2b^2}{a^2-4b^2}}{a^2+2ab+b^2}$,

111. $\frac{4x^2 + 4xy + y^2}{\frac{2x+y}{x^2-y^2} + \frac{6x+3y}{x^2+2xy+y^2}}$.

III. — Equazioni di 1° grado ad una incognita

Verificare se i valori indicati delle x sono soluzioni delle equazioni seguenti:

1. $\frac{12x-2}{6} - \frac{18-4x}{3} = x + 2$, $x = 3\frac{4}{7}$.
2. $5x - \frac{2x-1}{3} + 1 = 3x + \frac{x+2}{2} + 7$, $x = 8$.
3. $\frac{x+6}{4} - \frac{16-3x}{12} = \frac{25}{6}$, $x = 8$.
4. $\frac{7x+5}{3} - \frac{16+4x}{5} + 6 = \frac{3x+9}{2}$, $x = 1$.
5. $\frac{x+7}{3} - 5\frac{3}{4} = \frac{2x+5}{7} + \frac{10-5x}{8}$, $x = 8$.



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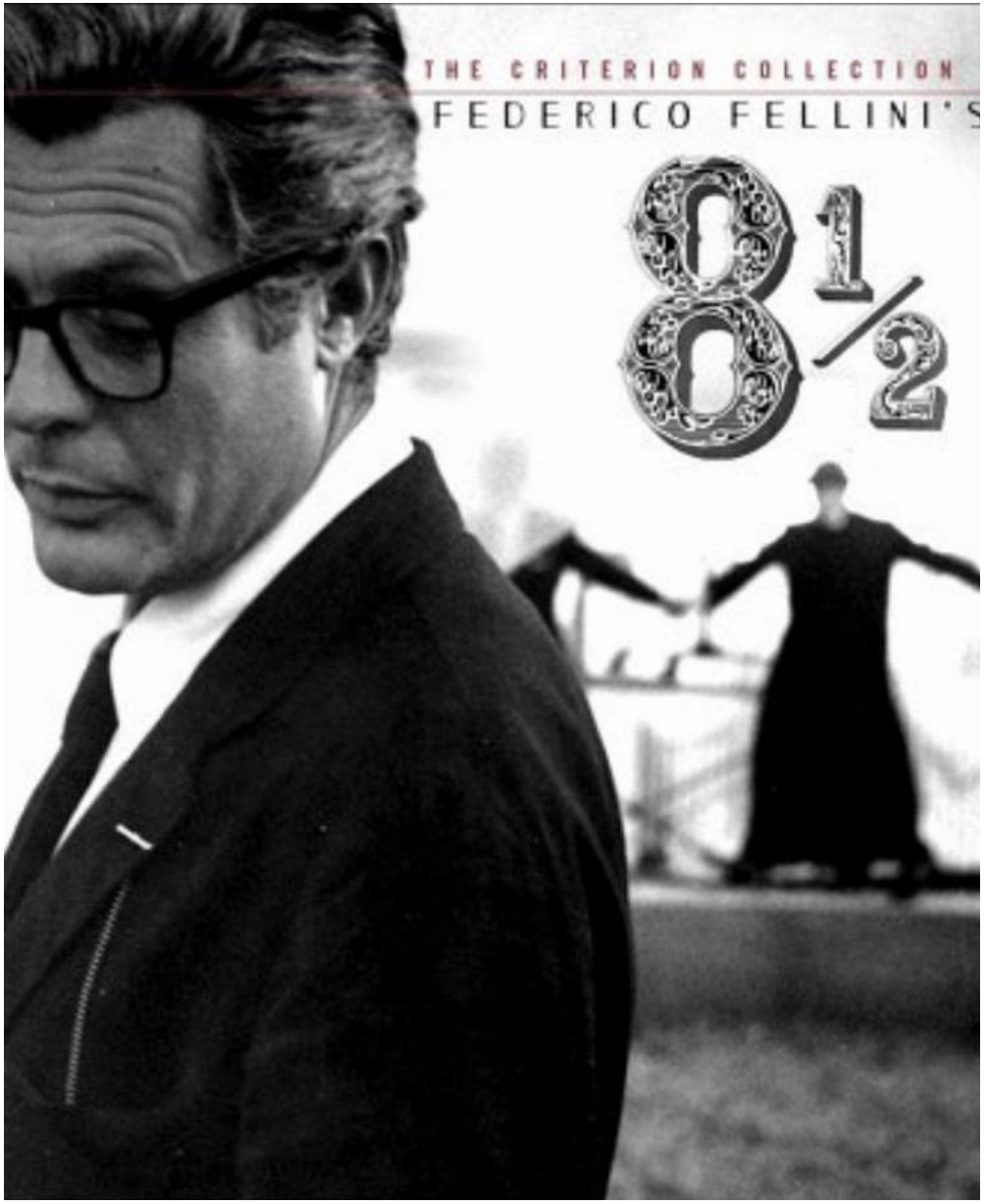
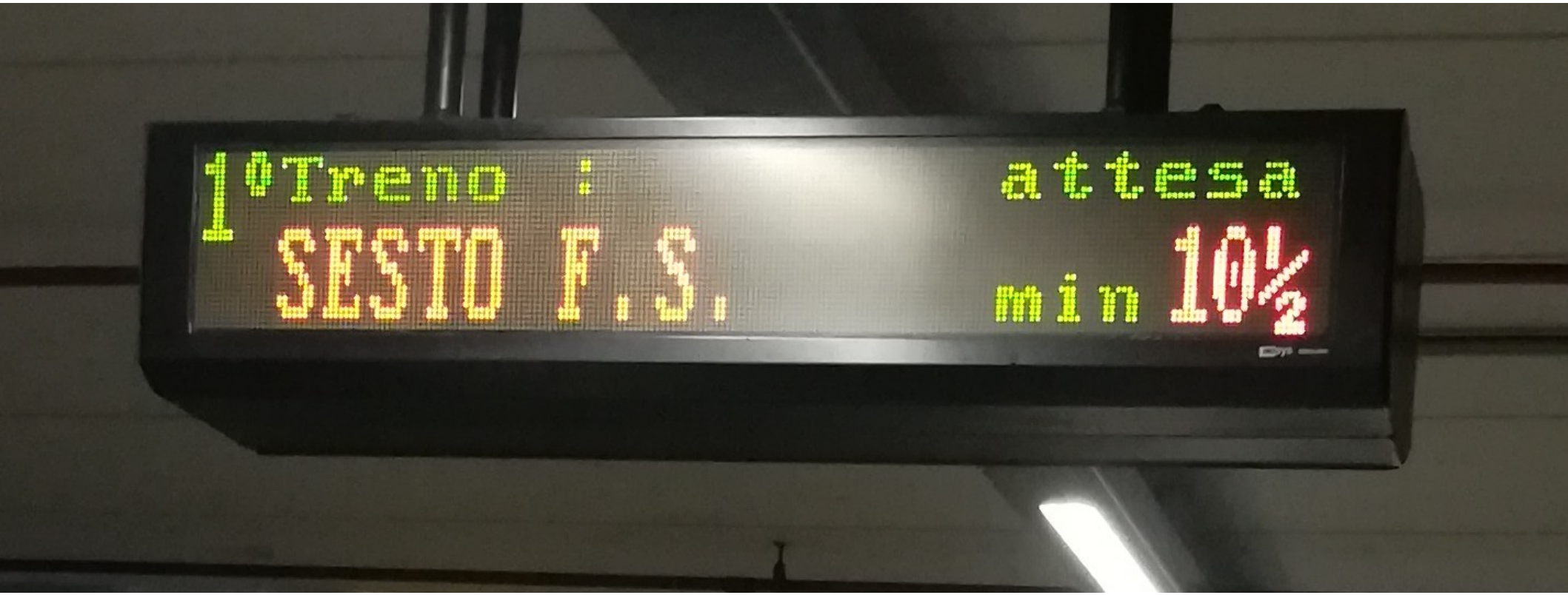
6. $\frac{2x+7}{7} - \frac{9x-8}{11} = \frac{x-11}{2}$, $x = 7$.
7. $\frac{3x-1}{7} + \frac{6-x}{4} - \frac{2x-4}{12} = 2 - \frac{x+2}{28}$, $x = 5$.
8. $4(x-3) - 7(x-4) = 6 - x$, $x = 5$.
9. $20x - 50(3x-4) = 200 - 20(4x-5) - (3x-4)50$, $x = 3$.
10. $10\left(x + \frac{1}{2}\right) - 6x\left(\frac{1}{x} - \frac{1}{3}\right) = 23$, $x = 2$.
11. $\frac{1}{3}(2x-10) - \frac{1}{11}(3x-40) = 15 - \frac{1}{5}(57-x)$, $x = 17$.
12. $\frac{1}{7}\left(x - \frac{1}{2}\right) - \frac{1}{5}\left(\frac{2}{3} - x\right) = \frac{1}{6}$, $x = 1\frac{1}{12}$.
13. $3x - 4 - \frac{4}{5}\frac{7x-9}{3} = \frac{4}{5}\left(6 + \frac{x-1}{3}\right)$, $x = 7\frac{1}{13}$.
14. $3\frac{1}{3}\left\{28 - \left(\frac{x}{8} + 24\right)\right\} = 3\frac{1}{2}\left(2\frac{1}{3} + \frac{x}{4}\right)$, $x = 4$.
15. $\frac{10x+17}{18} - \frac{12x+2}{11x-8} = \frac{5x-4}{9}$, $x = 4$.
16. $\frac{6x+18}{15} - \frac{3x+30}{5x-25} = \frac{2x}{5}$, $x = 20$.
17. $\frac{4x}{5-x} - \frac{20-4x}{x} = \frac{15}{x}$, $x = 3\frac{2}{11}$.
18. $\frac{5x+3}{x-1} + \frac{4x-3}{x-2} = 9$, $x = 1\frac{8}{13}$.
19. $\frac{30+6x}{x+1} + \frac{60+8x}{x+3} = 14 + \frac{48}{x+1}$, $x = 3$.
20. $\frac{1}{x-1} - \frac{2}{x+7} = \frac{1}{7(x-1)}$, $x = 7$.
21. $\frac{x}{x+1} - \frac{3x}{x+2} = -2$, $x = -\frac{4}{5}$.
22. $\frac{6x+8}{2x+1} - \frac{2x+38}{x+12} = 1$, $x = 2$.

But they can be seen in the wild



But they can be seen in the wild

But they can be seen in the wild



But they can be seen in the wild

But they can be seen in the wild

But they can be seen in the wild

shoes

recipes

However! Other reactions

- What these mean is so obvious there is no need to explain it.
- These are so confusing they should never be shown to anybody.

Linguistic issues

“Three and a half pizzas”

“Tre pizze e mezza”

You can say “A pizza and a half” in English but “Three pizzas and a half” would sound odd.

In Italian to force both parts of number together

need to ask “How many pizzas?” “Three and a half”

However! Other reactions

- You give me an excuse to burn some of my colleagues in middle school alive.
- I have students who arrive with this stuff in their heads from middle school
- I learned this in 2019 when my children went to middle school.

However! Other reactions

(Secondary school teacher with many years of experience)

- I can assure you with almost total confidence that people don’t do mixed numbers in maths lessons in Italy.

(...next day…)

- I asked my nephew. He did them in middle school last year.

Blame Finland?

- “Contaci”, middle school maths textbook published by Zanichelli in 1998, translated from Finnish, has mixed numbers because Finland is a mixed country. Is used in at least some middle schools in Italy.
- Several people claim this book is responsible for the reappearance of mixed numbers in Italy.
- Not clear how true this is.

But it gets more complicated

Trasforma le frazioni improprie in numeri misti.

ESEMPIO

Per trasformare una frazione impropria, ad esempio $\frac{27}{5}$, in numero misto si procede in questo modo:

- si divide il numeratore per il denominatore,

$$27 : 5 = 5 \quad \text{con resto } 2$$

- il quoziente è la parte intera del numero misto, il resto è il numeratore della parte frazionaria; quindi:

$$\boxed{\frac{27}{5} = 5 + \frac{2}{5}}$$

Middle school

Quite a few middle school teachers say mixed numbers exist but you need to put a plus in them.

- I asked if someone could send me an example of this notation being used in the wild. No.
- I asked about negative mixed numbers. Do they have an invisible minus in them?

Reactions include:

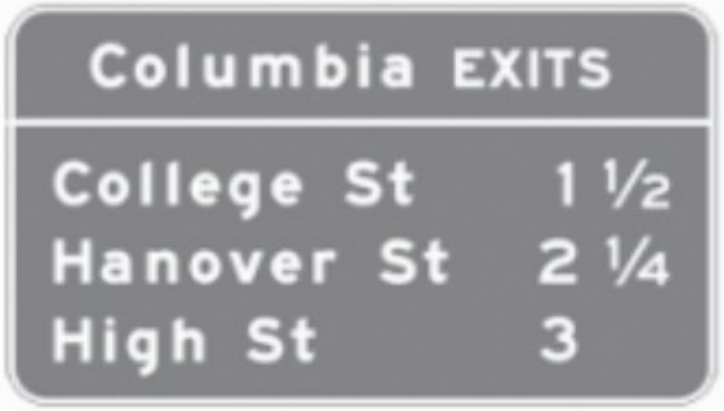
- They don’t exist
- -1¼ would mean -3/4 because of the invisiplus
- Maybe you’d write -(1+¼) or -1-¼ or -2+¾
- I’ve never thought about it
- We do negative numbers after we do mixed numbers, so it doesn’t come up.

EXAM QUESTION CONTROVERSY!

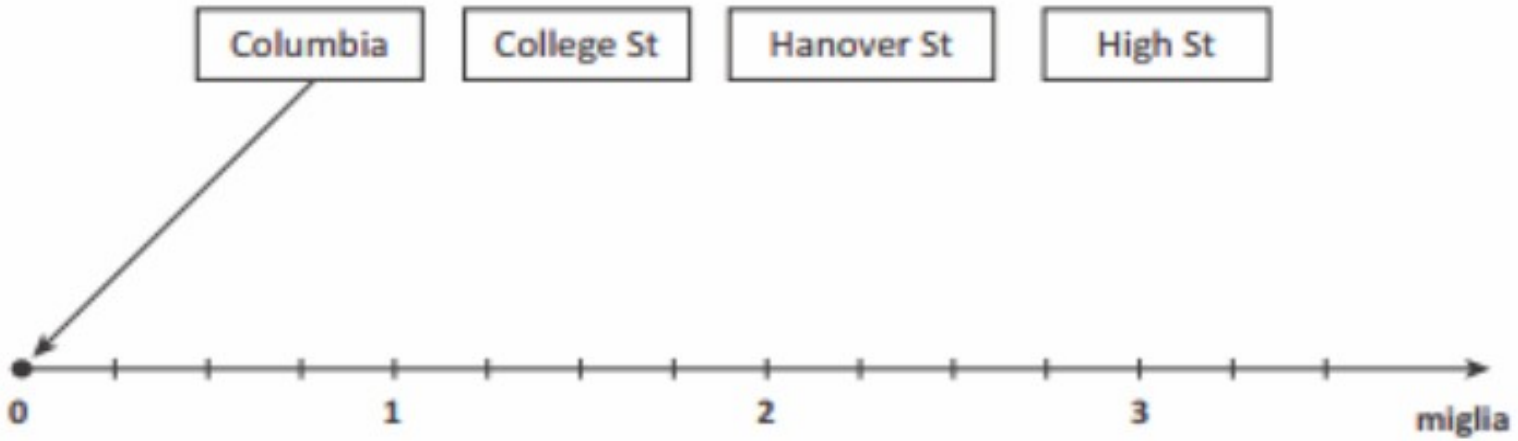
The following is from an exam sat by about 40 thousand 14 year olds a few years ago.

Only non-anecdotal thing in entire talk.

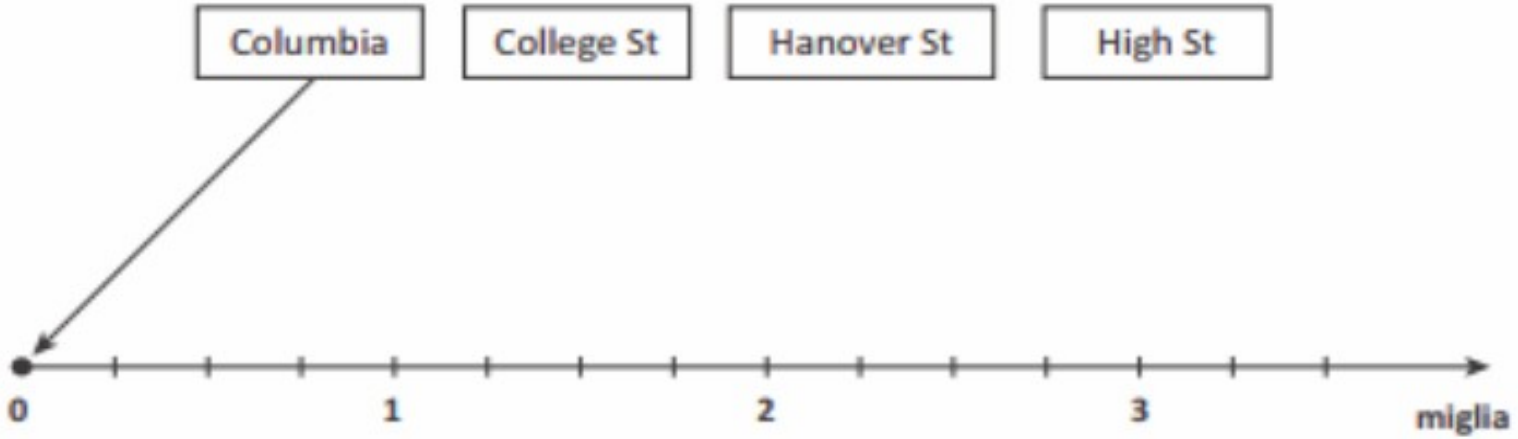
D4. **In figura viene riportato un cartello stradale americano che indica le distanze (in miglia) di tre località disposte lungo la stessa strada dall'uscita Columbia. Ad esempio, la distanza 1 ½ corrisponde a $1 + \frac{1}{2}$ miglia.**



a. **Collega con una freccia i riquadri corrispondenti alle località con la loro posizione sulla strada.**



a. **Collega con una freccia i riquadri corrispondenti alle località con la loro posizione sulla strada.**



b. **John esce all'uscita Columbia e vuole raggiungere College St. Se viaggia alla velocità media di 15 miglia all'ora, quanto tempo impiega?**

A. ☐ 6 minuti

B. ☐ 9 minuti

C. ☐ 12 minuti

D. ☐ 15 minuti

Prova	Anno	Liv.	Nr.	Domanda	✔	✖	☐	item
Mat - PN	2017	08	04 - A		54.4%	42.0%	3.6%	Mostra
Mat - PN	2017	08	04 - B		44.6%	49.4%	6.0%	Mostra

Do mixed numbers exist in your country?

- Yes
- No
- Yes, but not the way you write them

It gets worse

- Yes
- No
- Yes, but not the way you write them
- Yes, but they *aren’t numbers*

Some tell me

- Mixed numbers are not numbers. They are names/labels. You cannot perform arithmetic with them.
- Writing $\sqrt{2\frac{2}{3}}$

  is as meaningless as writing, say

$\sqrt{\text{medium rare}}$ or $\sqrt{\text{extra small}}$

Some tell me

- You cannot / do not perform any arithmetic operations on them.

  Maybe true for train platform numbers. But not persuasive for every single example.

  In particular, marks in school in Italy can be and are written as e.g. 6 ½. And average marks are a thing.

Yet others say

- They’re like greengrocers’ apostrophes – they’re a mistake made by people who don’t know any better.
- I know perfectly well what they are but they are an abomination. If I saw one in a scientific or technical context I would pretend not to understand it.

Nephew homework question

What is $2+\frac{1}{4}$?

Answer in book: $\frac{9}{4}$

No mixed numbers anywhere in his textbooks I could see

Conclusion?

I class Italy as unmixed because existence of multiple maths teachers and PhDs who say they've never seen them or that they are not numbers would be very peculiar in e.g. UK/USA.

(Some?) Middle school teachers seem to be a bit of an exception. But no-one else seems to be aware that they believe in mixed numbers.

Pisa Open Day people say "Oh that's American" not "Oh I saw that at middle school".

And an answer from elsewhere

- Yes but they are explicitly taught as being something found in foreign media/products.

Disclaimers etc.

I am not saying Italy should have or teach mixed numbers. That's very much up to Italy. I'm not trying to make fun of anyone.

It is clearly just not true that mixed numbers do not exist in Italy. They exist, and are used and understood at least in some contexts. It's possible that they have survived as traditional usages in those contexts even though elsewhere mixed numbers have vanished. And if it's true that mixed numbers are not taught in maths lessons any more then maybe people regard these usages as not maths or not numbers. Of course it's not even true that they are not taught in maths lessons. Whatever some teachers might say.

My suspicion is that even in the UK and US they are actually a fairly marginal topic. I don't know how much time is spent on them but it could easily be too much. It certainly seems to be the case that beyond a certain age at least in maths lessons one is probably encouraged to stop using them. Other than in combinatorial game theory, I would be somewhat surprised to see a mixed number in "actual maths" but they are definitely found in "real life". Perhaps more in the UK/USA than in Italy.

It was news to me, and apparently to some Italians, that mixed numbers exist in at least some middle schools. The "blame Finland" explanation may or may not be real but several people have offered it to me. I have to wonder if any awareness of mixed numbers persists after middle school – from conversations with potential students at Pisa University I am inclined to think it does not. However it's possible that mixed numbers are in the process of being reintroduced in Italy because of their existence in at least some middle school textbooks. Maybe if I gave my 1990 talk now things would be different. Not <u>everyone</u> would call me a liar, perhaps. Let's see what happens in another 10 years. Some people have told me they did mixed numbers in the 60s or 70s so not all Finland's fault.

It "matters" to the extent that if you (mostly English-speaking audience) give talks to international audiences and use mixed numbers some people in your audience might imagine that they are products, or that they are meaningless. Or, apparently, want to burn you alive. There is some risk of seriously disconcerting people, at any rate.

And part of the point of this talk is to show how hard it is to investigate something an issue like this. I lived in Italy for 5 years and have visited it regularly for about another 25 and I don't really understand quite what the situation is. Asking a small number of Italians "Do mixed numbers exist, yes or no?" would not have told me anything very useful. The answer is clearly more complicated than that.

I have used Italy as my example purely because I go there regularly and have access to lots of Italians. My best information is that Spain, France and Portugal are also "unmixed". However, I have no knowledge of the extent to which mixed numbers may exist in some contexts there and not others. I have asked two French people about mixed numbers and both said they have never seen them. But possibly I need to try harder to find a range of French people of differing ages and professions, and ask about special contexts.

For what it's worth, the youngest Italians I have access to do not seem to have done mixed numbers at school. But they are a very small sample. One book I've been shown had one entire line on mixed numbers in it so maybe some people blinked.