Sums of powers

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$$S_1 = \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

$$S_2 = \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

$$S_3 = \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4} = S_1^2$$

Sum of natural numbers: triangular numbers

Sum of squares: square pyramidal numbers

Sum of cubes

Nicomachus's theorem

$$S_3 = S_1^2$$

$$\sum_{k=1}^{n} k^3 = \left(\sum_{k=1}^{n} k\right)^2$$

e.g. $1^3 + 2^3 + 3^3 + 4^3 = (1+2+3+4)^2$

c. 100 A.D.

Cannonball problem: Are any of S_2 square?

$$S_1 = \sum_{k=1}^{n} k = \frac{n(n+1)}{2} = \frac{1}{2}(n^2 + n)$$

$$S_2 = \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} = \frac{1}{6}(2n^3 + 3n^2 + n)$$

$$S_3 = \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4} = \frac{1}{4}(n^4 + 2n^3 + n^2)$$

How to work out the next one?

Consider $\sum_{k=1}^{\mathrm{n}}(k^5 - [k-1]^5)$

$$n^5 = \sum_{i=1}^{n}(5k^4 - 10k^3 + 10k^2 - 5k + 1)$$

$$= 5\sum_{k=1}^{n} k^4 - 10\frac{(n^4 + 2n^3 + n^2)}{4} + 10\frac{(2n^3 + 3n^2 + n)}{6} - 10\frac{(n^2 + n)}{2} + n$$

$$\therefore \sum_{k=1}^{n} k^4 = \frac{1}{30}(6n^5 + 15n^4 + 10n^3 - n)$$

$$S_1 = \sum_{k=1}^{n} k = \frac{n(n+1)}{2} = \frac{1}{2}(n^2 + n)$$

$$S_2 = \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} = \frac{1}{6}(2n^3 + 3n^2 + n)$$

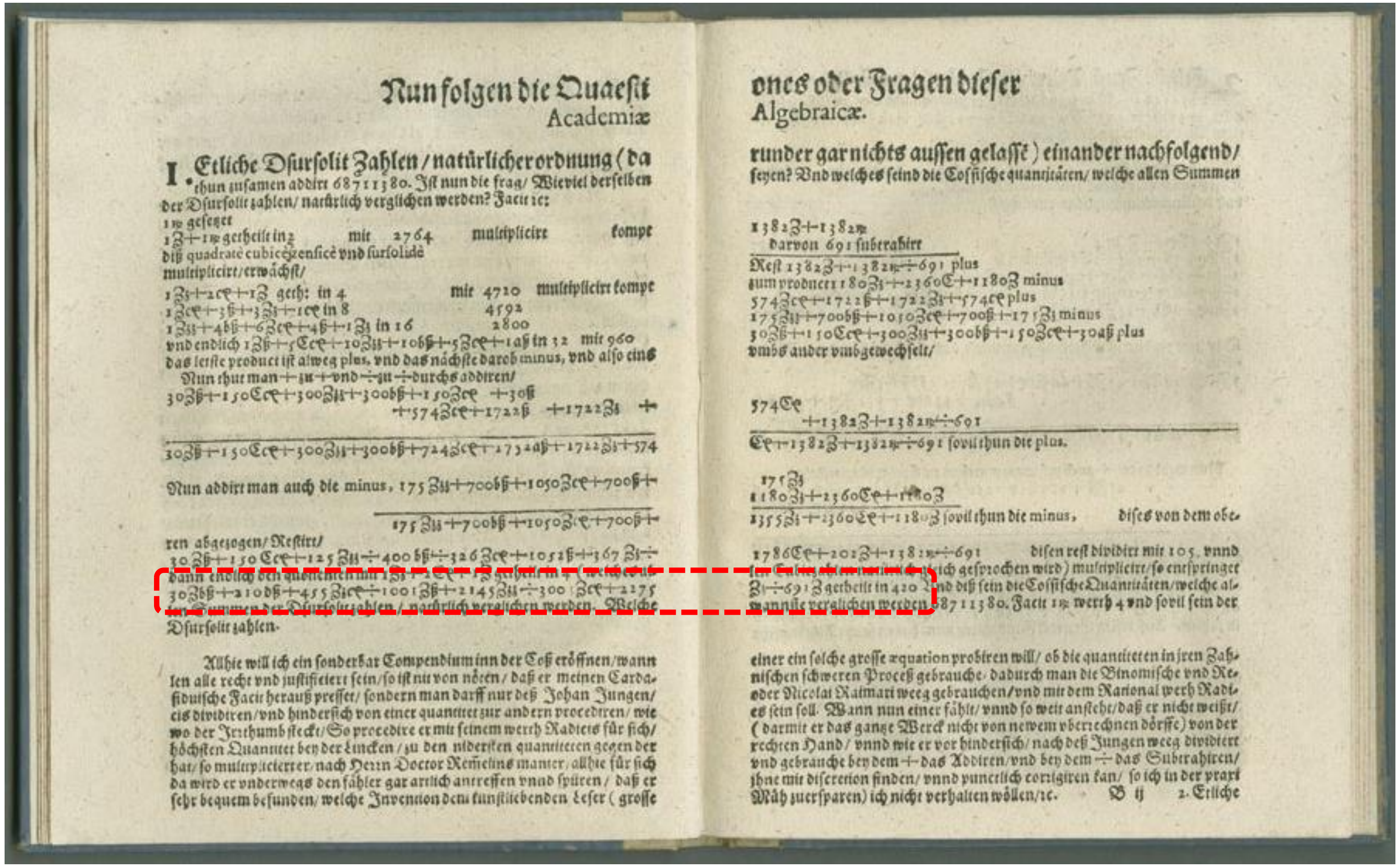
$$S_3 = \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4} = \frac{1}{4}(n^4 + 2n^3 + n^2)$$

$$S_4 = \sum_{k=1}^{n} k^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30} = \frac{1}{30}(6n^5 + 15n^4 + 10n^3 - n)$$

We can iterate to get subsequent ones.

Johann Faulhaber - The Great Arithmetician of Ulm

Academia Algebrae, 1631



$$\sum_{k=1}^{n} k^{13} = \frac{1}{420}(30n^{14} + 210n^{13} + 455n^{12} - 1001n^{10} + 2145n^{8} - 3003n^{6} + 2275n^{4} - 691n^{2})$$

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

1 8 28 56 70 56 28 8 1

1 9 36 84 126 126 84 36 9 1

1 10 45 120 210 252 210 120 45 10 1

1 11 55 165 330 462 462 330 165 55 11 1

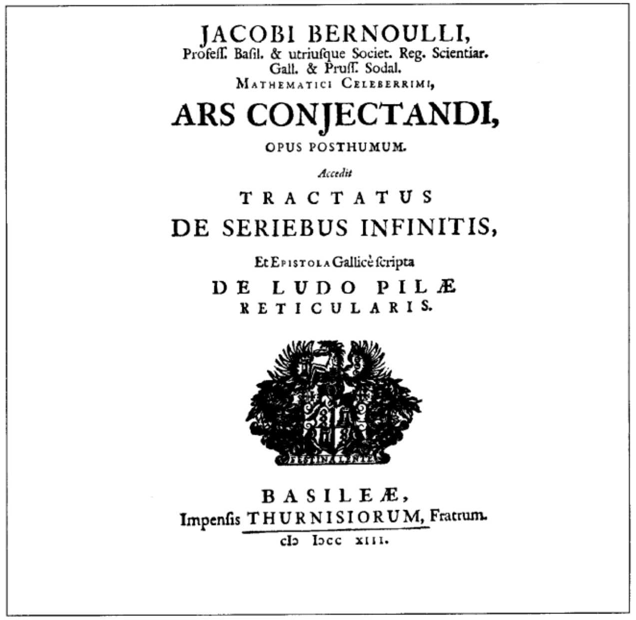
1											
1	0 1	0	0	0	0	0	0	0	0	0	
1	2	0 1	0	0	0	0	0	0	0	0	
1	3	3	0 1	0	0	0	0	0	0	0	
1	4	6	4	0 1	0	0	0	0	0	0	
1	5	10	10	5	0 1	0	0	0	0	0	
1	6	15	20	15	6	0 1	0	0	0	0	
1	7	21	35	35	21	7	0 1	0	0	0	
1	8	28	56	70	56	28	8	0 1	0	0	
1	9	36	84	126	126	84	36	9	0 1	0	
1	10	45	120	210	252	210	120	45	10	0 1	
1	11	55	165	330	462	462	330	165	55	11	1

1	0	0	0	0	0	0	0	0	0	0
-1	2	0	0	0	0	0	0	0	0	0
1	-3	3	0	0	0	0	0	0	0	0
-1	4	-6	4	0	0	0	0	0	0	0
1	-5	10	-10	5	0	0	0	0	0	0
-1	6	-15	20	-15	6	0	0	0	0	0
1	-7	21	-35	35	-21	7	0	0	0	0
-1	8	-28	56	70	56	-28	8	0	0	0
1	-9	36	-84	126	-126	84	-36	9	0	0
-1	10	-45	120	-210	252	-210	120	-45	10	0
1	-11	55	-165	330	-462	462	-330	165	-55	11

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -3 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -6 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -5 & 10 & -10 & 5 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 6 & -15 & 20 & -15 & 6 & 0 & 0 & 0 & 0 & 0 \\ 1 & -7 & 21 & -35 & 35 & -21 & 7 & 0 & 0 & 0 & 0 \\ -1 & 8 & -28 & 56 & -70 & 56 & -28 & 8 & 0 & 0 & 0 \\ 1 & -9 & 36 & -84 & 126 & -126 & 84 & -36 & 9 & 0 & 0 \\ -1 & 10 & -45 & 120 & -210 & 252 & -210 & 120 & -45 & 10 & 0 \\ 1 & -11 & 55 & -165 & 330 & -462 & 462 & -330 & 165 & -55 & 11 \end{pmatrix}^{-1}$$

$$
\begin{pmatrix} \sum 1 \\ \sum k \\ \sum k^2 \\ \sum k^3 \\ \sum k^4 \\ \sum k^5 \\ \sum k^6 \\ \sum k^7 \\ \sum k^8 \\ \sum k^9 \\ \sum k^{10} \end{pmatrix}
=
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1/2 & 1/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1/6 & 1/2 & 1/3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1/4 & 1/2 & 1/4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1/30 & 0 & 1/3 & 1/2 & 1/5 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1/12 & 0 & 5/12 & 1/2 & 1/6 & 0 & 0 & 0 & 0 & 0 \\
1/42 & 0 & -1/6 & 0 & 1/2 & 1/2 & 1/7 & 0 & 0 & 0 & 0 \\
0 & 1/12 & 0 & -7/24 & 0 & 7/12 & 1/2 & 1/8 & 0 & 0 & 0 \\
-1/30 & 0 & 2/9 & 0 & -7/15 & 0 & 2/3 & 1/2 & 1/9 & 0 & 0 \\
0 & -3/20 & 0 & 1/2 & 0 & -7/10 & 0 & 3/4 & 1/2 & 1/10 & 0 \\
5/66 & 0 & -1/2 & 0 & 1 & 0 & -1 & 0 & 5/6 & 1/2 & 1/11
\end{pmatrix}
\begin{pmatrix} n \\ n^2 \\ n^3 \\ n^4 \\ n^5 \\ n^6 \\ n^7 \\ n^8 \\ n^9 \\ n^{10} \\ n^{11} \end{pmatrix}
$$

Jakob Bernoulli
Ars Conjectandi, 1713



JACOBI BERNOULLI,
Profeff. Bafil. & utriufque Societ. Reg. Scientiar. Gall. & Pruff. Sodal.
MATHEMATICI CELEBERRIMI,
ARS CONJECTANDI,
OPUS POSTHUMUM.
Accedit
TRACTATUS
DE SERIEBUS INFINITIS,
Et EPISTOLA Gallicè fcripta
DE LUDO PILÆ
RETICULARIS.
BASILEÆ,
Impenfis THURNISIORUM, Fratrum.
cIↃ IↃcc XIII.

... Atque si porrò ad altiores gradatim potestates pergere, levique negotio sequentem adornare laterculum licet :

Summae Potestatum

$\int n = \frac{1}{2}nn + \frac{1}{2}n$

$\int nn = \frac{1}{3}n^3 + \frac{1}{2}nn + \frac{1}{6}n$

$\int n^3 = \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}nn$

$\int n^4 = \frac{1}{5}n^5 + \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n$

$\int n^5 = \frac{1}{6}n^6 + \frac{1}{2}n^5 + \frac{5}{12}n^4 - \frac{1}{12}nn$

$\int n^6 = \frac{1}{7}n^7 + \frac{1}{2}n^6 + \frac{1}{2}n^5 - \frac{1}{6}n^3 + \frac{1}{42}n$

$\int n^7 = \frac{1}{8}n^8 + \frac{1}{2}n^7 + \frac{7}{12}n^6 - \frac{7}{24}n^4 + \frac{1}{12}nn$

$\int n^8 = \frac{1}{9}n^9 + \frac{1}{2}n^8 + \frac{2}{3}n^7 - \frac{7}{15}n^5 + \frac{2}{9}n^3 - \frac{1}{30}n$

$\int n^9 = \frac{1}{10}n^{10} + \frac{1}{2}n^9 + \frac{3}{4}n^8 - \frac{7}{10}n^6 + \frac{1}{2}n^4$ ~~$-\frac{1}{12}nn$~~ $-{}^{3}/_{20}nn$

$\int n^{10} = \frac{1}{11}n^{11} + \frac{1}{2}n^{10} + \frac{5}{6}n^9 - 1n^7 + 1n^5 - \frac{1}{2}n^3 + \frac{5}{66}n$

Quin imò qui legem progressionis inibi attentuis ensperexit, eundem etiam continuare poterit absque his ratiociniorum ambabimus : Sumtâ enim c pro potestatis cujuslibet exponente, fit summa omnium n^c seu

$$\int n^c = \frac{1}{c+1}n^{c+1} + \frac{1}{2}n^c + \frac{c}{2}An^{c-1} + \frac{c\cdot c-1\cdot c-2}{2\cdot 3\cdot 4}Bn^{c-3}$$

$$+\frac{c\cdot c-1\cdot c-2\cdot c-3\cdot c-4}{2\cdot 3\cdot 4\cdot 5\cdot 6}Cn^{c-5}$$

$$+\frac{c\cdot c-1\cdot c-2\cdot c-3\cdot c-4\cdot c-5\cdot c-6}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8}Dn^{c-7}\ldots \text{\& ita deinceps,}$$

exponentem potestatis ipsius n continué minuendo binario, quosque perveniatur ad n vel nn. Literae capitales A, B, C, D & c. ordine denotant coëfficientes ultimorum terminorum pro $\int nn$, $\int n^4$, $\int n^6$, $\int n^8$, & c. nempe

$$A = \frac{1}{6}, B = -\frac{1}{30}, C = \frac{1}{42}, D = -\frac{1}{30}.$$

Bernoulli numbers

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} = \frac{1}{2}(n^2+n) = \frac{1}{2}\left(n^2 + \frac{1}{2}\binom{2}{1}n\right)$$

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6} = \frac{1}{6}(2n^3+3n^2+n) = \frac{1}{3}\left(n^3 + \frac{1}{2}\binom{3}{1}n^2 + \frac{1}{6}\binom{3}{2}n\right)$$

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4} = \frac{1}{4}(n^4+2n^3+n^2) = \frac{1}{4}\left(n^4 + \frac{1}{2}\binom{4}{1}n^3 + \frac{1}{6}\binom{4}{2}n^2 + 0\binom{4}{3}n\right)$$

$$\sum_{i=1}^{n} i^4 = \frac{n(n+1)(n^2+n+1)}{30} = \frac{1}{30}(6n^5+15n^4+10n^3-n) = \frac{1}{5}\left(n^5 + \frac{1}{2}\binom{5}{1}n^3 + \frac{1}{6}\binom{5}{2}n^3 + 0\binom{5}{3}n^2 - \frac{1}{30}\binom{5}{4}n\right)$$

…

$$\sum_{i=1}^{n} i^k = \frac{1}{k}\left(n^k + \frac{1}{2}\binom{k}{1}n^{k-1} + \frac{1}{6}\binom{k}{2}n^{k-2} + 0\binom{k}{3}n^{k-3} - \frac{1}{30}\binom{k}{4}n^{k-4} + 0\binom{k}{5}n^{k-5} + \cdots\right)$$

Bernoulli numbers: $1, \frac{1}{2}, \frac{1}{6}, 0, \frac{-1}{30}, 0, \frac{1}{42}, 0, \frac{-1}{30}, 0, \frac{5}{66}, 0, \frac{-691}{2730}, 0, \frac{7}{6}, 0, \frac{-3617}{510}, \ldots$

How can we calculate the Bernoulli numbers?

"I am doggedly attacking and sifting to the very bottom, all the ways of deducing the Bernoulli Numbers."
Ada Lovelace,
5 July 1843

$$B_8 = \frac{\begin{vmatrix} 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 3 & 0 & 0 & 0 & 0 & 0 \\ 1 & 4 & 6 & 4 & 0 & 0 & 0 & 0 \\ 1 & 5 & 10 & 10 & 5 & 0 & 0 & 0 \\ 1 & 6 & 15 & 20 & 15 & 6 & 0 & 0 \\ 1 & 7 & 21 & 35 & 35 & 21 & 7 & 0 \\ 1 & 8 & 28 & 56 & 70 & 56 & 28 & 8 \\ 1 & 9 & 36 & 84 & 126 & 126 & 84 & 36 \end{vmatrix}}{9!} = \frac{-1}{30}$$

Pascal's triangle again

Factorised

$$S_1 = \frac{1}{2}n(n+1)$$

$$S_2 = \frac{1}{6}n(n+1)(2n+1)$$

$$S_3 = \frac{1}{4}n^2(n+1)^2 = S_1^2$$

$$S_4 = \frac{1}{30}n(n+1)(2n+1)(3n^2+3n-1)$$

$$S_5 = \frac{1}{12}n^2(n+1)^2(2n^2+2n-1) = \frac{1}{3}(4S_1^3 - S_1^2)$$

$$S_6 = \frac{1}{42}n(n+1)(2n+1)(3n^4+6n^3-3n+1)$$

$$S_7 = \frac{1}{24}n^2(n+1)^2(3n^4+6n^3-n^2-4n+2) = \frac{1}{3}(6S_1^4 - 4S_1^3 + S_1^2)$$

$$S_8 = \frac{1}{90}n(n+1)(2n+1)(5n^6+15n^5+5n^4-15n^3-n^2+9n-3)$$

Faulhaber's theorem:

Sums of odd powers are polynomials in S_1.

Some nice relationships

$$S_3 = S_1^2$$

Nicomachus's theorem

$$S_5 + S_7 = 2S_1^4$$

References

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