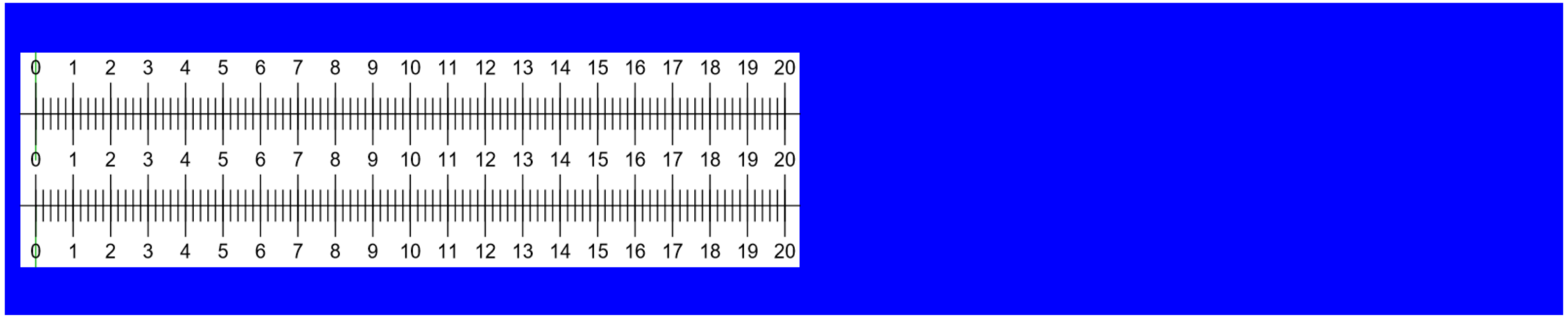
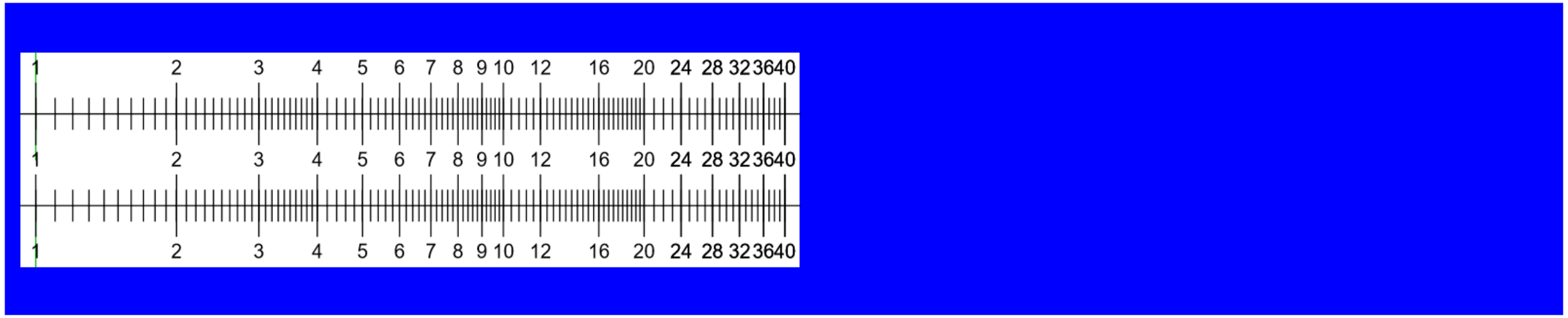
Sliderules II

The Kolmogorov Connection



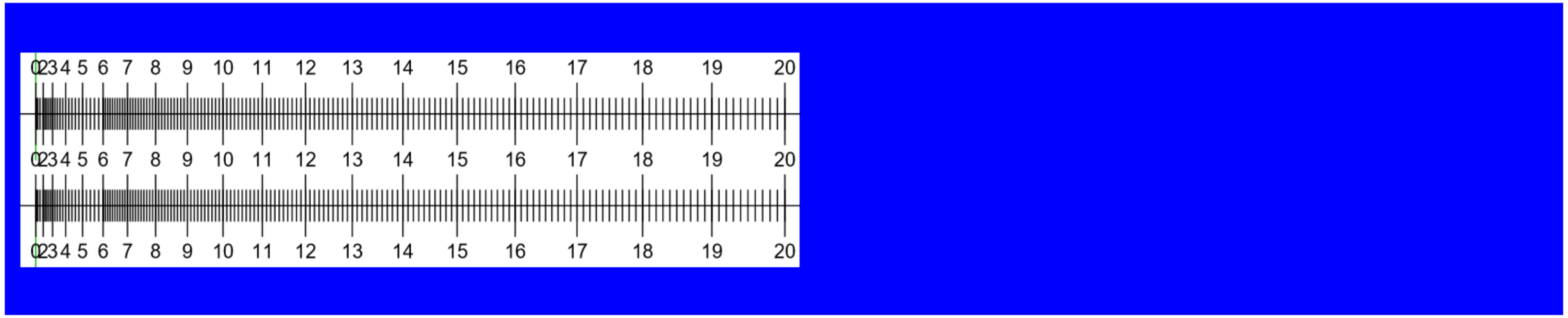
$$f(x) = x, \ f^{-1}(x) = x$$

$$S(x, y) = x + y$$



$$f(x) = \log(x),\ f^{-1}(x) = e^x$$

$$S(x, y) = e^{\log(x) + \log(y)} = x\ y$$



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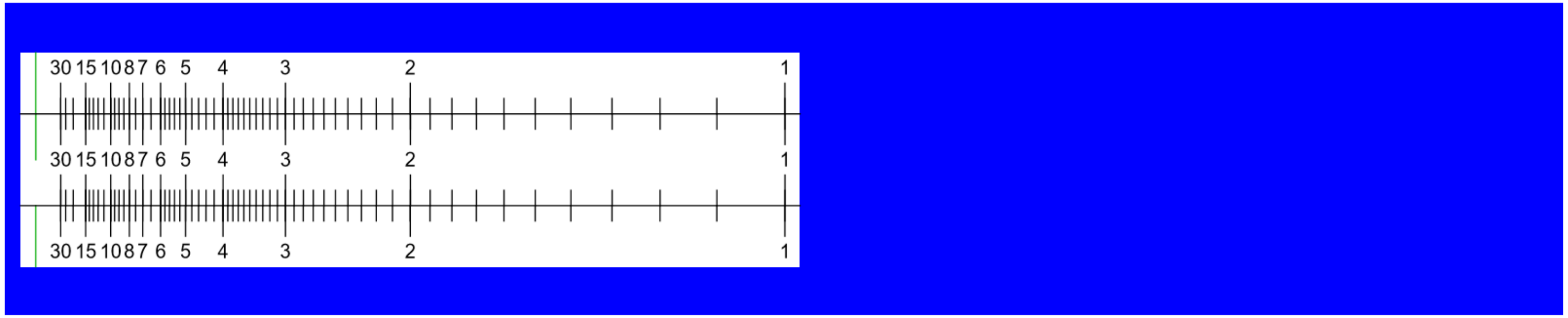

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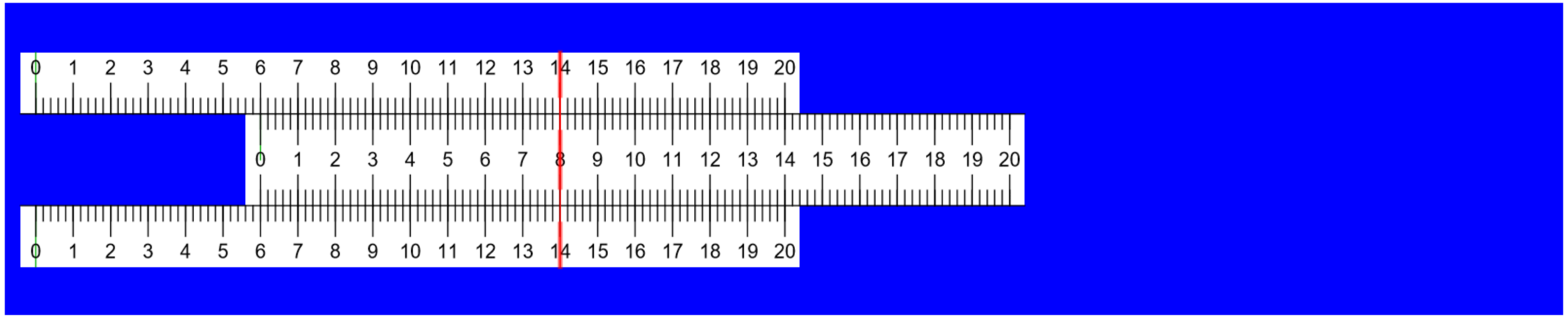
$$S(x, y) = \frac{1}{1/x + 1/y} = \frac{x\,y}{x + y}$$



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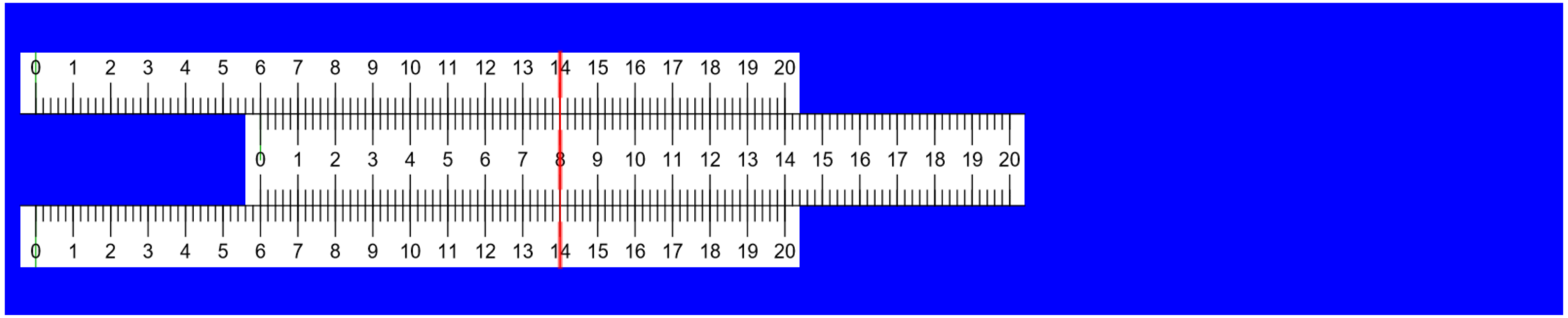

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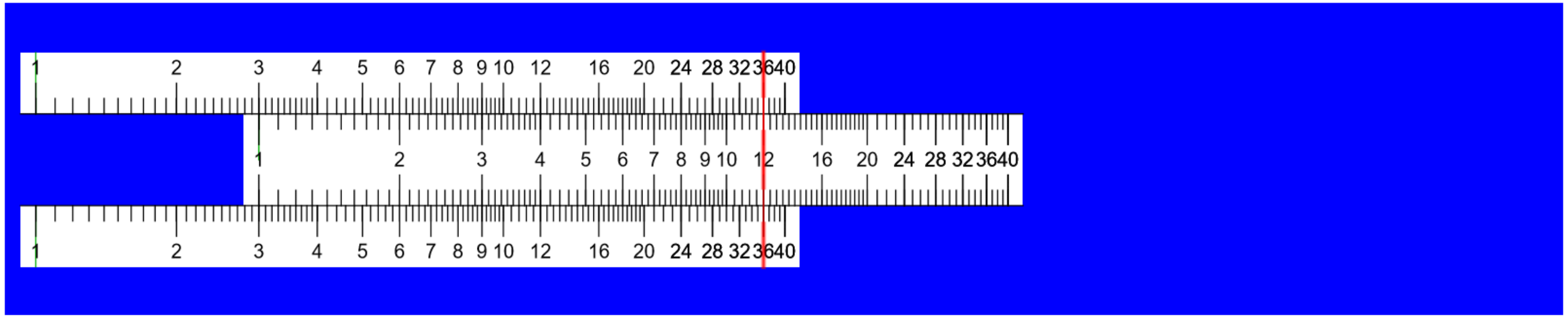

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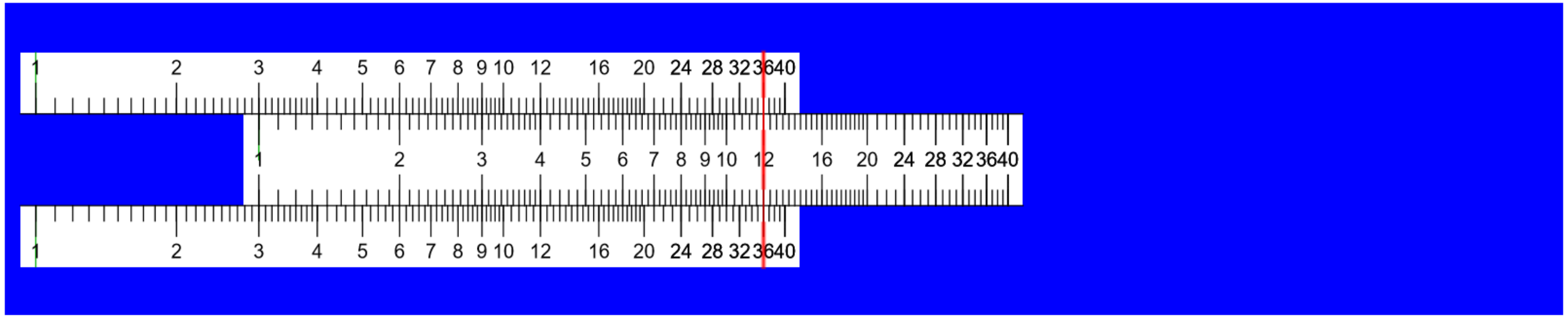

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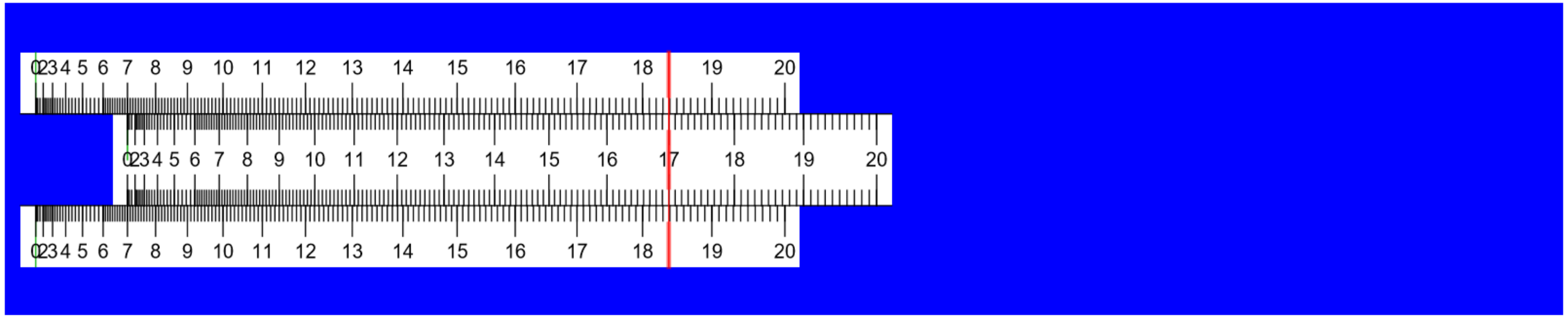

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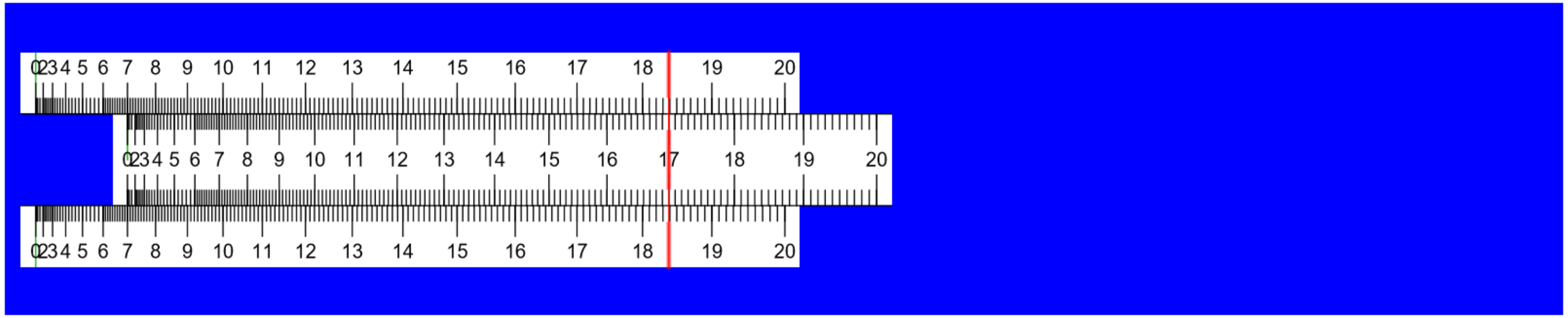

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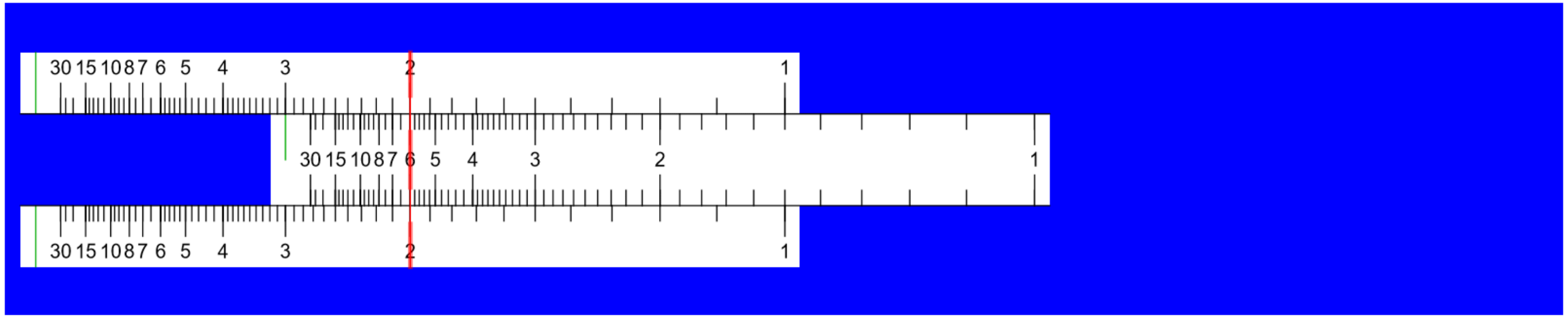

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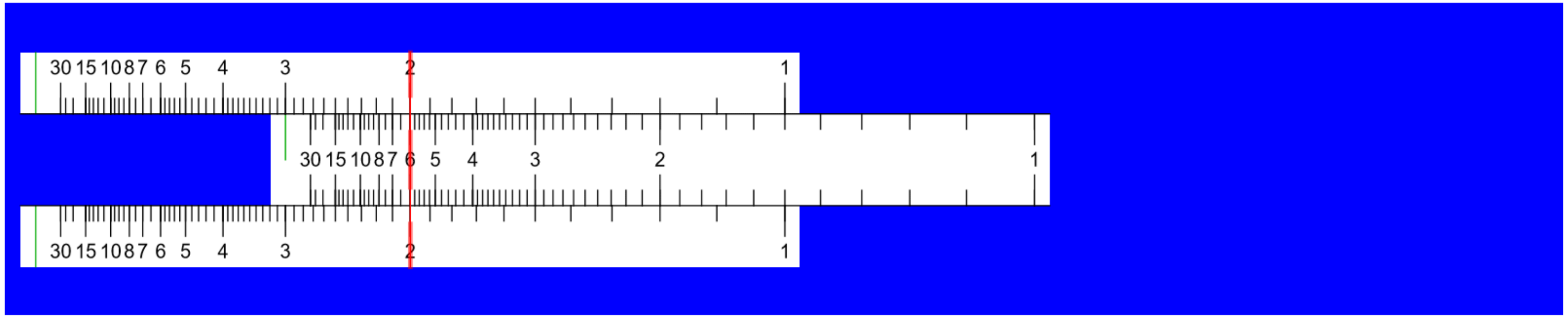

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It looks as if…

… doubling the spacing on the bottom scale makes a slide rule into…

… a mean machine.

(Sorry.)

What kinds of means?

Ordinary sliderule

$$S(x, y) = f^{-1}\big(f(x) + f(y)\big).$$

Double-spaced sliderule

$$M(x, y) = f^{-1}\left(\frac{f(x) + f(y)}{2}\right).$$

Generalised f-means.

Quasi-arithmetic means.

Kolmogorov means.

What I said in 2020

If $S(x,y) = f^{-1}(f(x) + f(y))$ (f continuous, monotonic, $f(e) = 0$) then:

- S is **continuous** and **increasing** in x and y;
- S is **symmetric**: $S(x,y) = S(y,x)$;
- S has an **identity**: $S(x,e) = x$;
- S is **associative**: $S\big(x, S(y,z)\big) = S(S(x,y),z)$.

(Easy to show.)

What I said in 2020

And conversely, given any S with those properties there exists an f such that $S(x, y) = f^{-1}(f(x) + f(y))$.

(Hard to show.)

What Kolmogorov said in 1930

If $M(x,y) = f^{-1}\left(\frac{f(x)+f(y)}{2}\right)$ (f continuous, monotonic) then:

- M is **continuous** and **increasing** in x and y;
- M is **symmetric**: $M(x,y) = M(y,x)$;
- M is **idempotent**: $M(x,x) = x$;
- M is **medial**: $M\big(M(w,x), M(y,z)\big) = M\big(M(w,z), M(y,x)\big)$

(Easy to show.)

What Kolmogorov said in 1930

And conversely, given any M with those properties there exists an f such that

$$M(x,y) = f^{-1}\left(\frac{f(x)+f(y)}{2}\right)$$

(Hard to show.)

Could I have piggy-backed on Kolmogorov?

Well, yeah. Though it's still some work.

Not obvious that **associativity** of S is implied by **mediality** of M, for example.

And where does that **identity** come from?

But yeah, I claim it can be made to work.

So, is that the whole story with means, then?

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I mean, no.

Desirable property: **homogeneity**:

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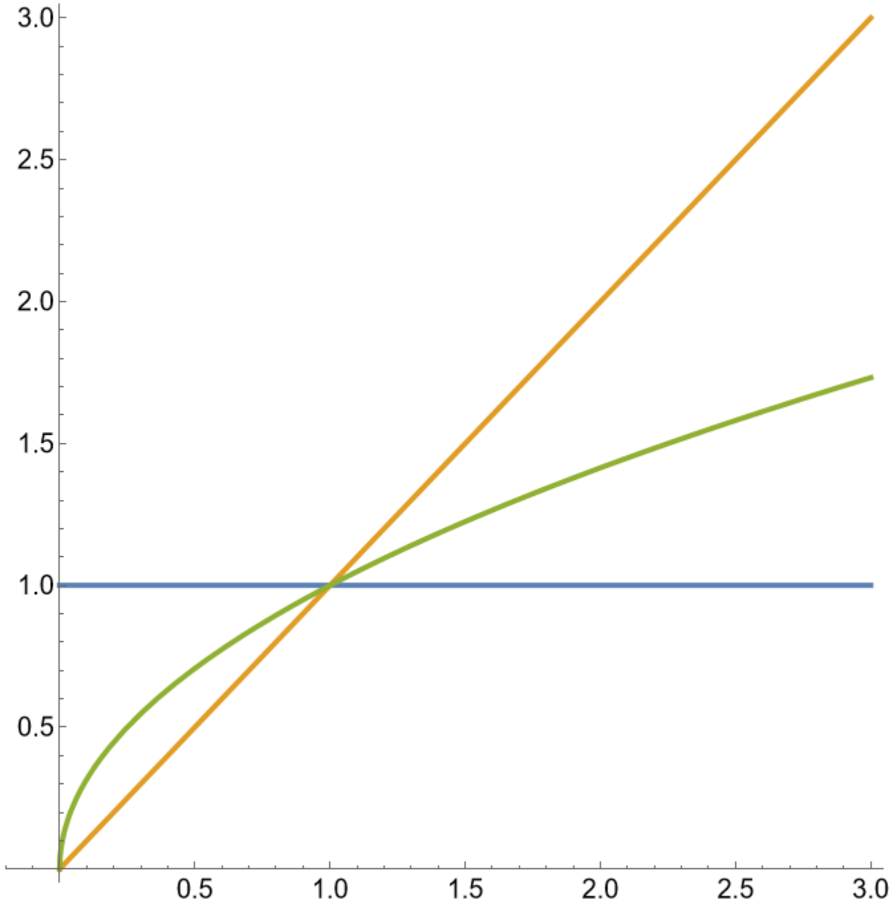

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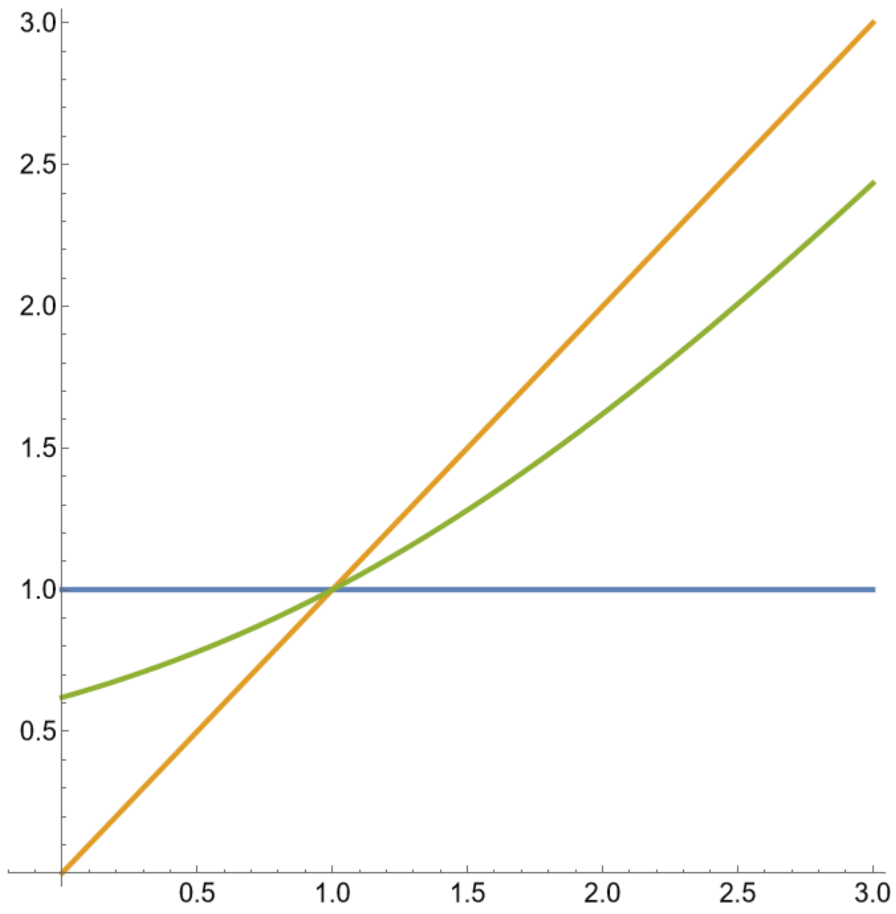
Desirable property: **homogeneity**:

$$M(ax, ay) = a\,M(x, y)$$

Most generalised f-means lack this property.



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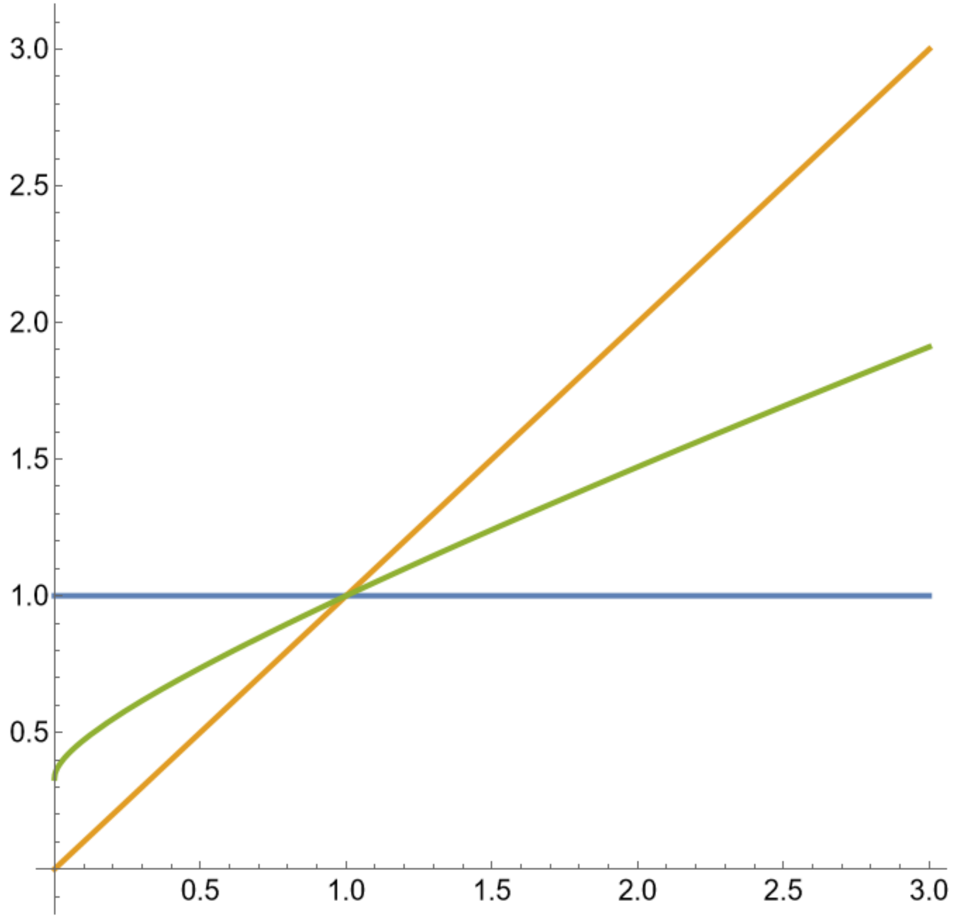
Example: **Heronian mean**

$$M(x,y) = \frac{x + y + \sqrt{x\,y}}{3}$$

Geometrical applications

Lacks **mediality** property.



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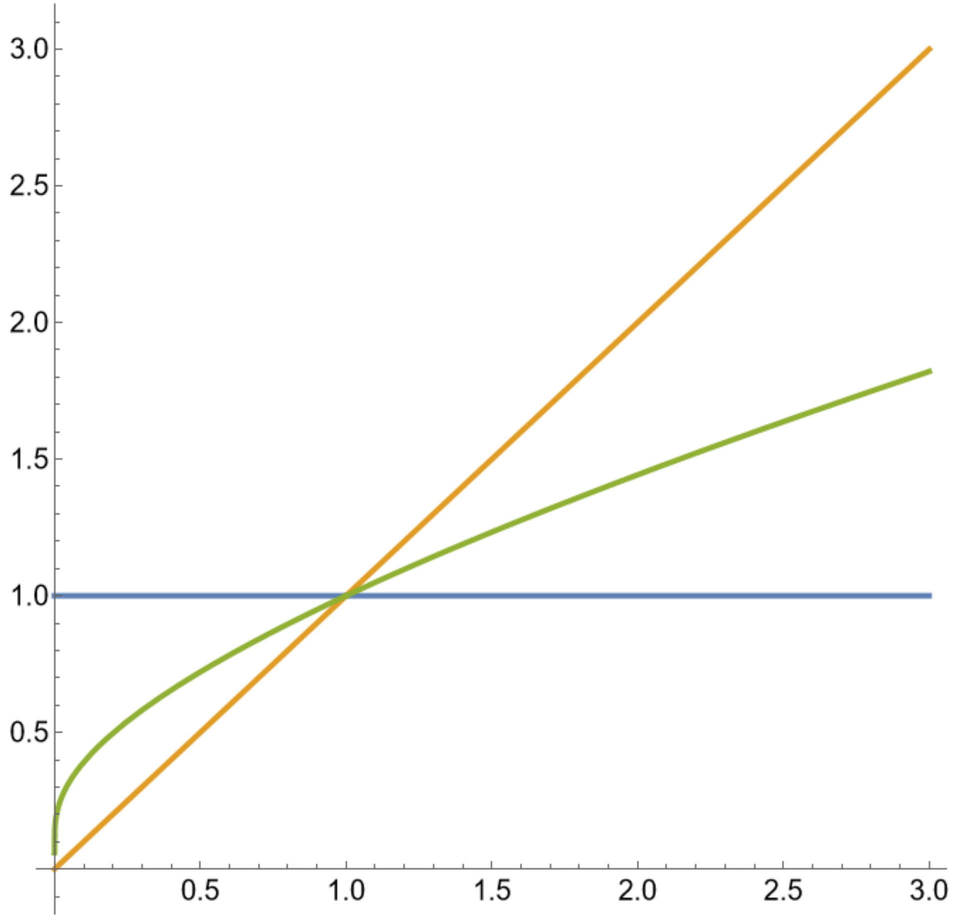
Example: **Logarithmic mean**

$$M(x,y) = \frac{x - y}{\ln x - \ln y}$$

Engineering applications

(Example of a **Stolarsky mean**.)

Lacks **mediality** property.



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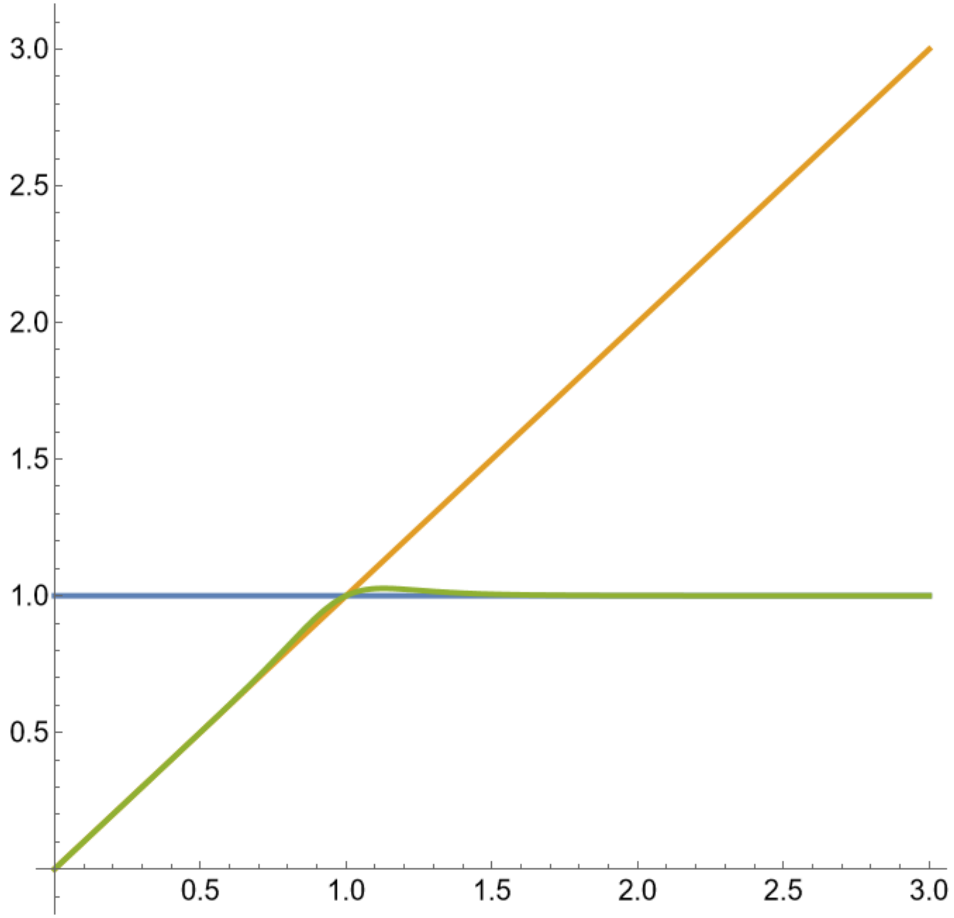
Example: **Lehmer p-mean**

$$M_p(x, y) = \frac{x^p + y^p}{x^{p-1} + y^{p-1}}$$

Signal processing applications

Medial for just two values of p.

Not even always **monotonic**!



What does “mean” even mean?

I MEAN, how long have we got?