9½-YEAR ANNIVERSARY SPECIAL

chalkdust

A magazine for the mathematically curious ISSUE 20 // AUTUMN 2024

Spirographs

Dear Dirichlet ✻ *Winning at Wythoff* ✻ *Ants* ✻
Complex triangles ✻ *Do you dress like a mathematical cliche?* ✻

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A magazine for the mathematically curious

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Somehow it's our 20th issue already, yet simultaneously fence posts v fence panels means that it's not also 10 years since we launched our first issue. How mildly irritating.

To celebrate, here are 20 of our top *Chalkdust* moments in the 9½ years so far:

- The Dear Dirichlet letter from a badger (issue 07)
- When Jane Hissey (author & illustrator of *Old Bear Stories*) let us use one of her pictures (issue 04)
- Running Taskmathster at MathsJam 2023
- The cover of issue 04
- Problem solving 101 (issue 04)
- The classifieds (issue 11)
- 🦂🦂🦂 (issue 03–)
- The draft letter for Dear Dirichlet issue 01 that was too rude to publish
- When we were the guest publication on *Have I Got News For You*
- The interview with Shannon Trust (issue 17)
- Issue 07
- Vhat Vhere Venn (issue 14)
- The Dear Dirichlet mug (merch.chalkdustmagazine.com)
- Hilbert's hotel: the board game (issue 08)
- When Bea taught us what whitespace was
- When Clare convinced us that we should be more Cosmo
- Not squaring the circle (issue 12)
- They might not be giants (issue 10)
- That time there was only space for 19/20 bullet points (issue 20)

The *Chalkdust* team

Acknowledgments

We would like to thank: all our authors and other contributors for writing and creating wonderful content; our sponsors for allowing us to continue making the magazine; our friends at the UCL department of mathematics; everyone at Achieve Fulfilment and Janet Townsend for their help with distribution. Thanks ants. Thants. The emojis used throughout this issue are Twemoji, licensed under CC BY 4.0.

ISSN 2059-3805 (Print). ISSN 2059-3813 (Online). Published by Chalkdust Magazine, Dept of Mathematics, UCL, Gower Street, London WC1E 6BT, UK.

IN CONVERSATION WITH

Robin Wilson

Ashleigh Wilcox and Ellen Jolley chat to the professor and maths communicator about books, the Open University and music

AS the author and editor of 55 books, with a career weaving through academia, teaching, and popular maths writing, professor Robin Wilson has a variety of interests. His books cover topics ranging from graph theory and the history of maths to sudoku, Lewis Carroll, and the comic operas of Gilbert and Sullivan. But he considers himself first and foremost a communicator of mathematics. "I've always been a populariser and a publicist for maths, and I've always been thought of more naturally as a teacher. But I've certainly been involved with research, particularly in graph theory and its history."

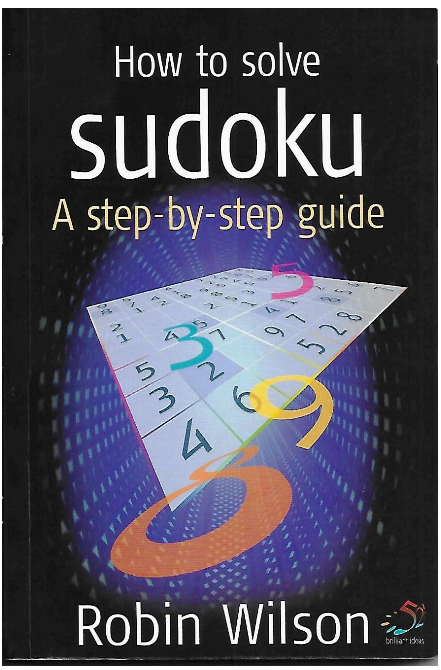
A prolific author

Many of his books lie in the areas of graph theory and combinatorics. He tells us how his first book, *Introduction to Graph Theory*, came to fruition. Still in print 52 years on, and now in its fifth edition, it began as a series of Oxford undergraduate lectures in graph theory. In those days, course lecture notes were sometimes in the form of booklets, purchasable from the university, so Robin wrote up his notes as a booklet, which was on sale for around £1. "At the time, there were few books in English on graph theory, and they were all too advanced and detailed, so you wouldn't find someone attending just eight lectures buying one of them." He continued: "I decided it would be useful to try and turn them into paperback book form." The first edition went on sale for £1.50. He has also written several other books on graph theory, and also on its history, of which he is an acknowledged expert.

One of Robin's most popular books is on *How to Solve Sudoku*, the now well-known combinatorial puzzle. He explained to us that "when sudoku puzzles came to this country in the autumn of 2004, and more and more newspapers started to include them, I enjoyed learning how to solve them. In mid-2005, there was a general election, and all the journalists were writing on political issues. But as soon as that was over, sudoku reached its peak, and all these journalists wanted to write about it, but didn't know what to write." A journalist approached Robin, calling him a sudoku expert, to which he replied, "No, I'm not an expert, I'm just an enthusiast." Robin also recalls that he was asked how sudoku arose and how to solve it, and replied informing the journalist that Euler had not invented it, in spite of the websites that claimed he did. The next day, Robin's paragraph appeared in the newspaper.

On the very same day, Robin received a phone call from a publisher inviting him to write a popular book on sudoku. Robin's initial instinct was to decline, but he then spent a few days researching and developing techniques for solving them. The publishers gave him three weeks to write the book, in a race against Carol Vorderman's book on sudoku. Robin rose to the challenge, writing it in just eleven days, and it was then published ten days later. Within just a few months it was available in twelve different languages. Robin recalls a highlight of the experience: "The BBC launched a puzzle magazine where my book was on the front cover of issue number one as a free gift. It gave me a real buzz, seeing it in the newspaper shops."

Many of his books, both on graph theory and on the history of mathematics, are co-edited. He has co-edited with Lowell Beineke (from Indiana, US) a set of six research-based books, *Topics in Graph Theory*, which was initially intended as only a trilogy. Robin explained to us how the aim is to turn research-level work into surveys that are sometimes more readable by graduate students. This involves writing to the top international researchers in a particular area, and collecting their contributions into a single volume in which the expositions are then made more readable where necessary, and where the organisation, terminology and notation are standardised throughout the book. Although "it's hard work being a good editor," Robin enjoys this role, and he continues to edit books, often because "they're books I've always wanted to have on my bookshelf."

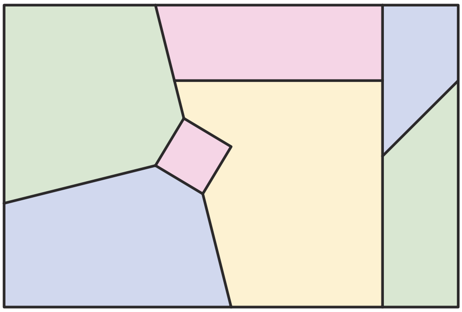


↑ One of Robin's popular books

He has also published revised editions of some of his written books. We asked Robin how new editions come about. "For my *Introduction to Graph Theory*, I initially produced a new edition every eight or ten years." Differences between editions sometimes arose in terminology; for example, "In the first edition I talked about circuits in graphs, but later 'cycles' became more standard." Other changes were improvements to the exposition, or revisions in the exercises and the inclusion of solutions. In the wake of breakthroughs, major changes needed to take place: "When the first edition appeared, the four colour problem was still unsolved, and later editions had to take account of its solution."

Robin next explained to us a project he worked on for several years. This project involved converting, into book form, an Open University course on the history of maths that was no longer being presented. He worked on this with two OU colleagues, the historians Jeremy Gray and June Barrow-Green. Volume 1 was published in 2019, and volume 2 in 2022, and Robin jokes, "They're wonderful for door stops; volume 1 is 500 pages and volume 2 is 700 pages."

These books present a source-based approach to the history of mathematics, with readers working through original sources, translated where necessary. Robin then went into more detail on some challenges of this project: "In many places we had to improve the exposition, and in order to make it more multicultural, we introduced more



↑ The four colour theorem states that any map can be coloured so that no two bordering countries are the same colour with at most four colours

Indian and Chinese mathematics and strengthened our treatment of Islamic mathematics. We basically reworked the entire course, incorporating the source material where necessary. My role was mainly (but not entirely) on the editorial side, while June and Jeremy reworked the historical content."

The university of the air

Robin's books on graph theory had their origins in his teaching in various universities. After completing his undergraduate degree at the University of Oxford, he went to the United States for his graduate work, being granted a PhD from the University of Pennsylvania. For his initial one-year scholarship, "the English-Speaking Union didn't just want someone who was going to get a first. They wanted someone with other interests, and I've always had a great interest in music." At Pennsylvania, he was then offered the chance to stay on as a teaching fellow: "I taught some calculus courses and thoroughly enjoyed this, and by the time I'd finished my PhD, I'd already presented 150 lectures, which provided me with a lot of teaching experience."

Back in the UK, he held positions at Cambridge and Oxford universities, and his interest in teaching led him eventually to the Open University, where he was a member of the mathematics faculty for 37 years. The OU had recently been founded as an innovative new model for university education. Originally envisioned as a 'university of the air', students with any level of qualification could earn a university degree through distance learning, with specially designed course materials mailed to students and TV and radio programmes broadcast by the BBC. In addition to remote learning, OU students also attended tutorial classes and summer schools, and it was at the OU's first maths summer schools in 1971 that Robin first became involved.

"I arrived basically sympathetic to the OU and its maths foundation course, and had two weeks teaching at an OU summer school in north Wales. I came back wildly enthusiastic. The students, many of whom had no previous mathematical background, worked from 9 am to 9 pm, and then went to the bar for a drink where they chatted about linear algebra. I don't recall any Oxford students chatting in the bar about linear algebra!"

Robin's experience at the OU summer schools was so positive that he decided to apply for a longer-term job there, and succeeded: "I was getting very enthusiastic about the whole idea. I was really thinking, this is my place." He joined the OU full-time just as the first pure maths course was being developed. "I found it very exciting, and quite daunting, not least because I had to make five BBC nationally broadcast television programmes in my first three months, in addition to the appropriate training for this. I think that I was always quite a competent presenter, but I was never a natural; I eventually presented around 30 TV programmes, but was always much happier with my radio broadcasts."

The TV programmes were originally recorded at Alexandra Palace in north London, the former home of the BBC which had been recently vacated with their studios available for use. "Studio A was the larger one where we usually did the recording, and studio B was the smaller one where we had our rehearsals. We once had to record in studio B, and when I remarked that it seemed rather cramped, the producer said:

'Don't criticise: in the 1950s we once broadcast Verdi's *Aida* live from here!' We usually had rehearsals about a week beforehand, and recording was then a two-day job of incessant rehearsals followed by a one-hour recording session when we had to get everything right."

Not everyone had such a positive opinion about the Open University. In particular, "the academic world was sceptical, saying that 'You can't get a degree just by watching the telly!'" In fact, it was the specially written correspondence texts that provided the main teaching resource, and the television was partly to give visual backup for reading them. Robin enthused: "The original staff members were quite remarkable. They came not only from academia, but also from schools and industry, and were sufficiently motivated about the OU concept to give up their previous full-time jobs to try this crazy thing which might not work."

Mathematics and music

Robin has taken advantage of his position at the Open University to take courses in some of his other interests. He has always had a strong interest in music, and has been a choral singer since his teenage years, as well as taking part in many musical stage productions. "When I joined the OU, I decided to take advantage of its courses on the fundamentals and history of music, and eventually, combining these with other OU arts courses, I was able to complete a second BA degree, an honours degree in humanities with music."

Music remains a huge part of his life. He regularly performs in musical theatre and operetta. "I recall appearing in Sweeney Todd, when I had to have my throat cut, with blood everywhere! In the coming year, I hope to take part in stage productions of Rodgers and Hammerstein's *Cinderella*, and in *Jesus Christ Superstar*."

It's clear that, from graph theory to show business, Robin Wilson has a knack for finding harmony in all the things he loves.

↑ Robin appears as Euler in a history of mathematics lecture

Ashleigh Wilcox
Ashleigh is a PhD student and graduate teaching assistant at the University of Leicester. Her main mathematical interests are in number theory. She is passionate about outreach and inclusion in mathematics, volunteers as a Stem ambassador and is a co-lead of the Piscopia Initiative.

Ellen Jolley
Ellen is a mathematician based at the University of Warwick in Coventry. She is studying the mechanics of hydrogels (what soft contact lenses are made of) for their use in medical devices.

Weather this week: variable, uncontrolled

NEWS

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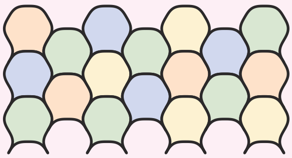
vvvvvvvvvvvvvvvvvvvvvvvvv

Biologists find a use for maths—our love is tainted!

Earlier this year, professors Alain Goriely, Gábor Domokos, Ákos G Horváth and Krisztina Regős published their discovery of a way to tile space without sharp corners: soft cells.



Mathematicians have long studied how shapes can fit together to cover surfaces without gaps. But the typical approach—using shapes with sharp corners and flat faces—is rarely seen in the natural world. Living organisms use a dazzling array of patterns to form and grow, as seen in muscle cells and layers of onion bulbs.

These patterns are just shapes with curved edges, non-flat faces and few sharp corners. How nature reaches these complex patterns using 'soft shapes' has eluded mathematical explanation…until now! The newly-discovered *soft cells* tile space with a minimal number of sharp corners. In 2D, these soft cells have curved boundaries with only two corners. Amazingly, in 3D, soft cells have no corners at all.

Soft cells are found abundantly in nature and seem to be the building blocks of biological tissue, and understanding this phenomenon could answer why certain patterns are preferred by nature.

Not only that, but they could also unlock new building designs, devoid of corners. We have hired the team to build *Chalkdust* HQ.

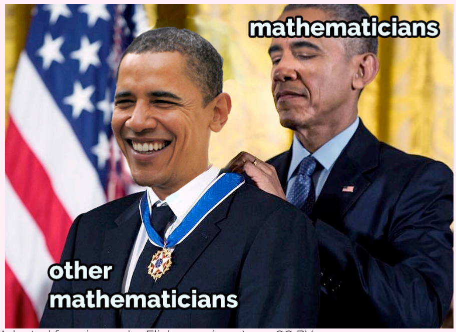
Beggar-my-*Chalkdust* editor

In May, we locked two of our editors in a room to test if the infinite game of beggar-my-neighbour really is infinite.

They haven't yet told us that the game is finished, so we consider it a success.

Mathematician wins Zeeman medal; unsurprising

The 2024 Christopher Zeeman medal is awarded to Brady Haran, of *Numberphile* fame.



Adapted from image by Flickr user jurvetson, CC BY 2.0

↑ Mathematicians giving mathematicians awards for being inspiring mathematicians

Every two years, the councils of the IMA and the London Mathematical Society huddle together to decide who should win the Christopher Zeeman medal, an award recognising the efforts made into maths communication across the globe.

Former BBC journalist Brady Haran seeks the world's most interesting mathematicians and interviews them, talking through various concepts in science in an all-accessible manner.

Brady has been publishing blogs, vlogs and interviews since 2011 through various YouTube channels: *Numberphile*, *SixtySymbols*, and *Periodic Videos*, to name a few. In this time he has accumulated over 1.3 billion views, with a watch time of 11,000 years, with *Numberphile* attaining nearly 700 million views on its own.

The Zeeman medal will follow his honorary doctorate and Medal of the Order of Australia for his efforts in science communication.

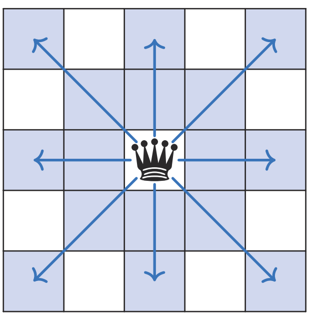
Brady's videos have encouraged young mathematicians to pursue careers in all things Stem. This includes, but is not limited to, writing news pages and cryptic crosswords for a popular maths magazine.

Winning Wythoff's game

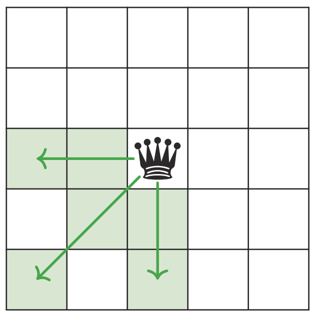
Molly Ireland presents a gambit which will impress your mates

ARE you looking for a new game to beat your friends at? If so, look no further! I'm going to show you a simple game—just two players and one piece on a board—where you will be able to beat your friends every time, meeting a well-known sequence on the way...

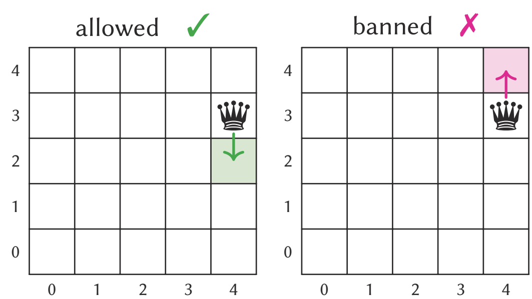
Let me introduce you to Wythoff's game. Named after the Dutch mathematician Willem Abraham Wythoff, it involves moving a queen on a two-dimensional board. If you've ever played chess, the movements of a queen will be familiar to you: she can move right, left, up, down and along any of the diagonals until she reaches the boundaries of the board:



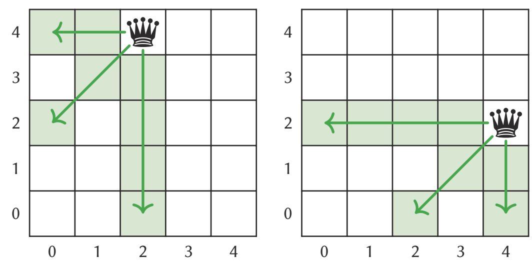
Here's the catch: in Wythoff's game, the queen can only have some of these moves. She must not move in any way right or up. This gives a reduced set of moves:



So for example, in Wythoff's game, a move from (4, 3) to (4, 2) is allowed but moving from (4, 3) to (4, 4) is banned:



Below are some examples of the moves available when the queen is in different positions on the board. In the diagrams, the queen is allowed to move to *any* position along the arrows:



On a larger board, the pattern is the same: we're always only interested in what's below and to the left of the queen.

How to play

At the beginning of the game the queen is assigned a starting position by both of the players—this could be done by randomly choosing a square on the board or by you each picking your favourite number to get a coordinate.

The rules of the game are as follows:

- Two players take it in turns to move the queen.
- On each turn, a player must move the queen to a different square.
- The player who cannot move the queen on their turn loses the game.

A player loses the game when the queen can't be moved due to the boundaries on the left-hand side and the bottom of the board. You can try it out and see that this only occurs at the square $(0,0)$. By ending a turn on $(0,0)$ we leave our opponents with no moves and win the game!

A little position analysis

Alright, so $(0,0)$ is the magic square to land on, but how do we make sure we get there before our opponent?

We start by grouping the squares into two groups: the helpful 'P' positions and the unhelpful 'N' positions:

- *N positions* are positions from which the *n*ext player to make a move will eventually win.
- *P positions* are positions from which the *p*revious player (who ended their turn by moving to the P position) will eventually win.

Here, $(0,0)$ is a very special P position as the player to move the queen to $(0,0)$ will win the game immediately. The positions which end the game are given the special name of *terminal positions*.

In our quest to get to $(0,0)$, we should think about the relationship between P and N positions. Let's define a *follower*, x', of a position, x, as any position that can be reached from x by a player in one turn. For example, $(0,0)$ is a follower of the position $(1,0)$, and is reached by moving the queen one position to the left.

We can now say something about the behaviour of P and N positions:

1. The terminal positions are P positions.
2. All the followers of P positions are N positions (since landing on a P position leads to a win, so the player starting from that position should lose).
3. Every N position has at least one follower which is a P position (which is how the next player will eventually win).

Starting from the terminal position $(0,0)$ and working backwards, we can categorise all the positions on the board of Wythoff's game in the following way:

Step 1: Begin at the terminal position $(0,0)$ and highlight it blue. This is a P position by (1):

5						
4						
3						
2						
1						
0	P					
	0	1	2	3	4	5

Step 2: Consider all the positions that have $(0,0)$ as a follower. From (2) we know all the followers of P positions are N positions. The positions we are considering all have a P position as a follower so must instead be N positions. Highlight these N positions orange:

5	N					N
4	N				N	
3	N			N		
2	N		N			
1	N	N				
0	P	N	N	N	N	N
	0	1	2	3	4	5

Step 3: Move to a position that has not yet been coloured in, but where all the followers of this position have. If all the followers of this position are N positions then by (2) and (3) this must be a P position. If at least one of the followers is a P position then by (2) the position is an N position:

5	N					N
4	N				N	
3	N			N		
2	N	P	N			
1	N	N	P			
0	P	N	N	N	N	N
	0	1	2	3	4	5

Step 4: Repeat the previous step until all the positions on the board have been coloured. Filling in the P and N positions of Wythoff's game for the board here then goes like this:

5	N					N
4	N				N	
3	N	N		N		
2	N	P	N			
1	N	N	P	N		
0	P	N	N	N	N	N
	0	1	2	3	4	5

↓

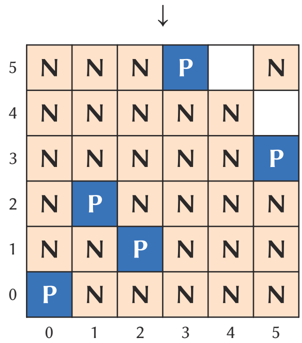
5	N					N
4	N	N			N	
3	N	N	N	N		
2	N	P	N	N		
1	N	N	P	N	N	
0	P	N	N	N	N	N
	0	1	2	3	4	5

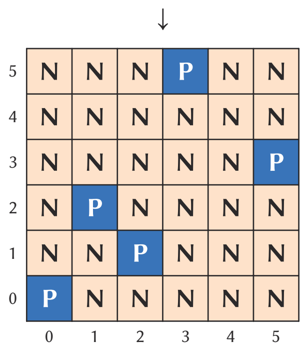
↓

5	N	N				N
4	N	N	N		N	
3	N	N	N	N		
2	N	P	N	N	N	
1	N	N	P	N	N	N
0	P	N	N	N	N	N
	0	1	2	3	4	5

↓

5	N	N	N			N
4	N	N	N	N	N	
3	N	N	N	N	N	
2	N	P	N	N	N	N
1	N	N	P	N	N	N
0	P	N	N	N	N	N
	0	1	2	3	4	5



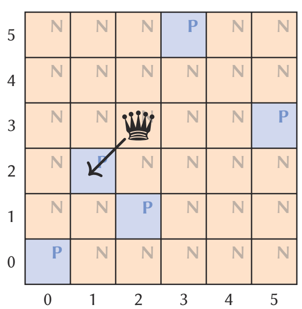


Tada! Once the P and N positions have been determined, the winning strategy is to always end your go by moving to a P position when possible. Using the P positions like stepping stones, you can reach the position $(0, 0)$ and win.

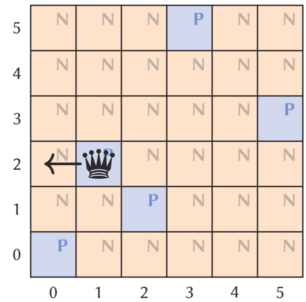
Strategy in action

Time for an example game! Let's consider the case where the queen begins at $(2, 3)$ and that we are the player to move the queen first.

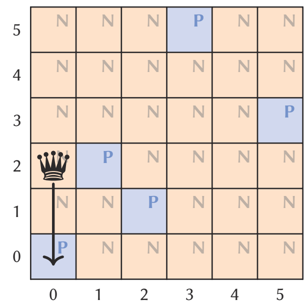
On our first move we move to the P position $(1, 2)$:



This leaves our opponent with only N positions to choose from, and they choose to move to $(0, 2)$:



We respond by moving to the terminal position $(0, 0)$, winning the game and presumably lots of money/drinks/respect:



Are we the champions?

Alright, so you've trapped a friend or lured in an innocent bystander to play with you; you are eager to try out your strategy. But before you get too confident, you've got to ask yourself one question: are you *guaranteed* to win (punk)?

If you haven't already noticed it, go back and see if you can spot the trick to this strategy. The guarantee is real... but you have to get to a P position before your opponent! This means we must be the first player if the queen starts on an N position, or the second player if the queen starts on a P position. Deciding who starts is then what determines the winner of the game if you both know the winning strategy.

If you want to stop your friend uncovering your winning strategy it might therefore be beneficial to deliberately make a few mistakes here and there, throwing them off the scent (or rip out these pages in *Chalkdust* before you let them borrow it).

With the colour-coded cheat sheet of the board we made earlier, you should be all set to go. But if lugging around a sheet of colourful paper does not appeal to you, it might interest you to take a look at the coordinates of the P positions. These are symmetric and the first of these pairs is $(1, 2)$ or equivalently $(2, 1)$. The second pair is at $(3, 5)$ or $(5, 3)$. Feel familiar?

It turns out that a subset of the P positions of the game are given by the positions corresponding to a pair of Fibonacci numbers! All the P positions of the game have coordinates of the form

$$(\lfloor n\phi \rfloor, \lfloor n\phi^2 \rfloor)$$

and the symmetric equivalent $(\lfloor n\phi^2 \rfloor, \lfloor n\phi \rfloor)$. Here, $\lfloor\cdot\rfloor$ is the floor function which will round the argument down to the nearest integer; and ϕ is the golden ratio, which is the number approached by the ratio of consecutive Fibonacci numbers as we tend to infinity. Here are the coordinates of the P positions for $0 \leqslant n \leqslant 7$:

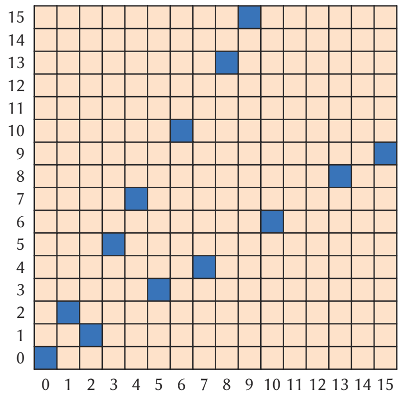
n	0	1	2	3	4	5	6	7
$\lfloor n\phi \rfloor$	0	1	3	4	6	8	9	11
$\lfloor n\phi^2 \rfloor$	0	2	5	7	10	13	15	18

This form of the P positions is given in Wythoff's discussion of the game from 1907, *A Modification of the Game of Nim*, which you can find and read online.

The proof of this is a rather long proof by induction which I'll let you try yourself, but essentially you have to show that these pairs satisfy the behaviour of the P and N positions set out in ①, ② and ③.

What's next?

A standard chess board is 8×8, but the terminal position of our game is only affected by the boundaries of the left and bottom edge of the board. If we extended the standard chess board to have more squares our strategy would remain unchanged. This means you can choose your board to be as large as you want—even infinite!—however you may want it to be a little smaller so that it fits on your coffee table. Here are the P positions on a 16×16 board:



If you're keen to see what happens when you play with multiple queens, check out the book *Winning Ways for Your Mathematical Plays*, volume 1, by Berlekamp, Conway and Guy. The analysis of P positions for many different games and the behaviour discussed at the bottom of page 9 is explored in Thomas Ferguson's book *A Course in Game Theory*. For further discussion on how Fibonacci numbers link to the P positions see *Around the World in 80 Games* by Marcus du Sautoy.

You now have the knowledge to beat your friends! Good luck—but if you decide the starting order, you won't need it.

Molly Ireland

Molly is a fourth-year mathematician at Durham University studying for her masters, where her favourite areas are mathematical biology, decision theory and, of course, game theory. She loves a puzzle: sudoku, cryptic crossword, … you name it! She is also working on crocheting miniatures of her family's dogs.

Do you dress like a mathematical cliche?

Fashion fiend or fashion flop? Take our quiz to find out!

1. What's your university profile picture like?

- 20 years old – and I still have that jumper!
- A photo of me on a mountain
- Me explaining some maths

2. Socks with sandals?

- Never! A cardinal fashion sin
- Comfy AND practical
- I prefer tights – mine have sums on

3. What do you wear to teach?

- Same as usual: tweeds and a tie
- Same as usual: fleece and cargo pants
- Same as usual: anything with maths on

4. What's the biggest fashion faux pas?

- Confusing a tie for a cravat
- Being caught without waterproofs
- No such thing!

5. Do you wear any mathematical jewellery?

- Just my signet ring
- My Garmin watch does loads of calculations
- I have SO MANY mathematical earrings

6. How do you carry things around?

- In a briefcase
- My backpack – it's got a built-in Camelbak
- I made this tote bag – look, it's actually a Klein bottle!

7. Any tattoos?

- No.
- *Take nothing but photos, leave nothing but footprints*
- Yeah, under here I've got Navier–Stokes…

8. When it gets cold, what are you throwing on?

- My trusty velvet smoking jacket
- A quarter-zip fleece: practical AND practical
- An *infinity* scarf (get it? get it??????)

Mostly
You're **tweedy trad.**

It's classic and timeless. The shirt–waistcoat–jacket combination will never go out of fashion (was it ever *in* fashion?!). You've been wearing the same clothes for decades – dark academia comes to YOU for inspiration.

Mostly
You're **hiker chic.**

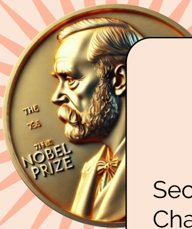
You're ready for anything the day throws at you, whether it's hiking in the rain or hiking in the sunshine. The pockets in your cargo shorts are full of chalk (for climbing AND for blackboards) and your calculator's on a carabiner. Mountains, here you come.

Mostly
You're **equationcore.**

You're all about communicating maths – and your clothes do it for you. You wear a ring-on-a-chain as a necklace, and you've got maths pun T-shirts for days. There's a lull in conversation? Has it been five minutes since you brought up your tattoos?

what's **hot** and what's **not**

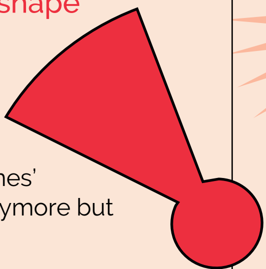
Maths is a fickle world. Stay à la mode with our guide to the latest trends.



▲ HOT

This shape

Behold! A square. Not sure why we've decided that the 'straight' part of 'straight lines' isn't important anymore but OK boomer.



~~~~~~

Hats and other tiling shapes

This winter, we're exclusively covering our bathrooms with keyhole shapes. *Who'd live in a house like this...?*

▼ NOT

▲ HOT

Being outraged at the Nobel prize

Secretly you are all apoplectic your ChatGPT blog post wasn't nominated.

~~~~~~

Writing papers which qualify

Getting your code to compile. That's it. That's the real Nobel prize.

▼ NOT

▲ HOT

Going on *More or Less* to talk about maths

Maths people agreeing agreeably with maths people. In terms of peak radio, it's not exactly *Johnnie Walker's Sounds of the Seventies*... but what is?

Going on GB News to talk about maths

But I guess the money's good, right? Right?

▼ NOT

▲ HOT

Overnight temperatures on 10 October

No need to worry about putting the heating on tonight. Thanks, BBC Weather!

~~~~~~

Sanity-checking your supplier's data before publishing it everywhere

The Met Office would never have given such dodgy data

▼ NOT

Agree? Disagree? Tag us on socials @chalkdustmag + more advice online

▲ HOT

-T-O-G-O

Mathematicians like a well-defined dance.

▲ HOT

Taking long exposures of the sky and telling your friends you saw the northern lights

Nothing to do with maths, it's just literally all our group chats right now.

~~~~~~

The *Northern Lights* trilogy

Did Mick Herron write it? No? Then I'm not reading it.

▼ NOT

A complex twist on triangles

Chris Sangwin uncovers the equation of a triangle using the power of complex numbers

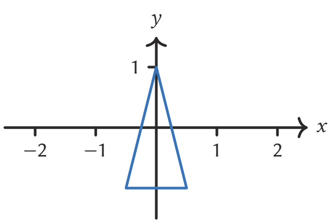
HAVE you ever wondered how to describe a triangle in the form of an equation? It's a question that seems deceptively simple. In two recent videos, Matt Parker dives into this geometric puzzle, challenging viewers with the following problem:

Given three points A, B and C in the plane, find an equation for the triangle with these points as vertices.

For certain triangles, this is straightforward; for example, if we plot the graph of

$$|x| + \big|y + |x|\big| = 1,$$

we get this triangle:



However, finding a general equation that will describe any triangle–not only those with vertices at convenient grid points–is harder to approach.

Many creative solutions emerged. Some used trigonometric functions, and modular arithmetic. One even took the tangent of a number, and then immediately took the arctangent of the result. However, none used the often-overlooked power of complex numbers. How can we tackle this problem with these elegant objects?

Complex numbers and implicit functions

The first question to clarify is: what does it mean to find the equation of a triangle? Our goal is to link the geometric triangle to an algebraic representation, in the form of an equation.

The French mathematician Jacques Hadamard is often quoted, "The shortest and best way between two truths of the real domain often passes through the imaginary one." That is to say, many results are easier to state and prove using complex numbers.

Rather than using two coordinates (x, y) to specify a point, we can use a single complex number z. As a trivial example, compare the real and complex equations for a circle. Points (x, y) lie on the circle centred at (a, b) with radius r if and only if

$$(x-a)^2 + (y-b)^2 = r^2.$$

Complex numbers z lie on the circle centred at p with radius r if and only if

$$|z-p| = r.$$

Since this is a considerably shorter equation, it is often described as being simpler.

A reasonable objection to these observations is that significant work is somehow hidden inside the formula. Inside $|z-p| = r$ is the definition of modulus, which relies on Pythagoras' theorem, and when unpacked this is identical to the Cartesian equation. All notation, to some extent, abbreviates difficult or lengthy ideas. It is exactly the same with vectors: the equation of a line in $\mathbb{R}^2$ is given by $\boldsymbol{x}\cdot\boldsymbol{n} = c$. This *could* be unpacked with coordinates as

$$\begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix} = c \iff ax + by = c.$$

We swap compactness of $\boldsymbol{x}\cdot\boldsymbol{n} = c$, in which points on the line (represented by $\boldsymbol{x}$) are treated as single objects, with the ability to look inside at the relationship between coordinates in $ax + by = c$.

Some more useful information can be extracted from the line equation if we introduce complex numbers. More specifically, by way of the complex conjugate $\bar{z} \coloneqq x - \mathrm{i}y$, then the equation of a straight line ℓ avoiding the origin in the complex plane can be written as

$$\bar{z}p + z\bar{p} = 2,$$

where $p \in \mathbb{C}$ is given. Here, a segment from the origin to p is perpendicular to the line ℓ, and the line ℓ is a distance $1/|p|$ from the origin.

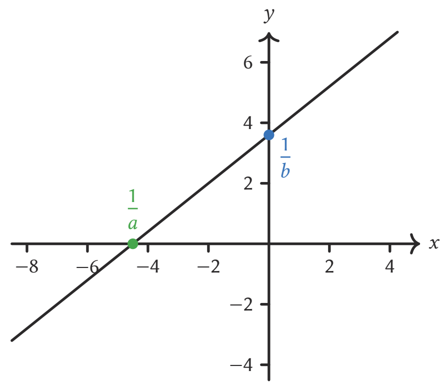
If $z = x + \mathrm{i}y$ and $p = a + \mathrm{i}b$ then $\bar{z}p + z\bar{p} = 2$ becomes

$$(x-\mathrm{i}y)(a+\mathrm{i}b) + (x+\mathrm{i}y)(a-\mathrm{i}b) = 2.$$

Expanding out, and dividing by two, gives us directly that

$$ax + by = 1,$$

which is the equation for a straight line avoiding the origin. This is illustrated below. The x-axis intercept of $ax + by = 1$ is when $y = 0$, ie when $x = 1/a$. The y-axis intercept of $ax + by = 1$ is when $x = 0$, ie when $y = 1/b$. Showing that a segment from the origin to $p = a + \mathrm{i}b$ is perpendicular to the line ℓ, and the line ℓ is a distance $1/|p|$ from the origin, is a fun exercise to try for yourself—or you can just take my word for it.



Equilateral triangles in the complex plane

Complex numbers can characterise geometric relationships in very succinct and elegant ways. For example, complex numbers u, v and w form the vertices of an equilateral triangle if and only if

$$u^2 + v^2 + w^2 = uv + vw + wu.$$

To establish this result, we need to undertake an *analysis* of the relationships between the three variables u, v and w. One strategy to start might be to show for fixed z that

$$u^2 + v^2 + w^2 = uv + vw + wu$$

if and only if

$$(u-z)^2+(v-z)^2+(w-z)^2 = (u-z)(v-z)+(v-z)(w-z)+(w-z)(u-z).$$

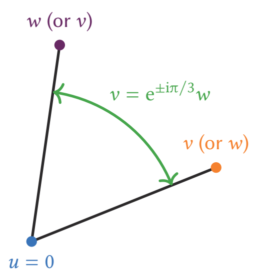
Introducing yet another variable z risks making the formulae appear even more complex. However, this proves that the identity is invariant under translation and so, without loss of generality, we can assume one point is at the origin, and set $u = 0$. Then we have a much simpler job with understanding the relationship between v and w when (setting $u = 0$)

$$v^2 + w^2 = vw.$$

Rearranging, we have $v^2 - vw + w^2 = 0$, which holds if and only if

$$v = \frac{1 \pm i\sqrt{3}}{2} w = e^{\pm i\pi/3} w.$$

Hence $|v| = |w|$ and the angle between v and w is $\pi/3 = 60°$. Interpreting the algebra geometrically, u (the origin), v and w must form an equilateral triangle.



This result is an 'if and only if', so it would be possible to start with three points in the plane, characterise in a geometric way when they form an equilateral triangle and then derive the equation. More generally, to find the equation of a triangle we actually want to build an equation with particular properties. Building something requires *synthesis*, which is a slightly different process. In particular, we shall break up the problem of creating the equation of a polygon into two steps:

1. Find the equation of a segment.
2. Combine equations of segments, each representing an edge, to an equation for the complete polygon.

In this case the second step, combining equations, is simple because we are combining them with a logical *or* operation. Points lie on one *or* another of the segments. Equations can be combined by taking a product of terms, iterating the observation that

$$p(x) = 0 \text{ or } q(x) = 0 \iff p(x)q(x) = 0.$$

For example, when solving a factored quadratic $(x-a)(x-b) = 0$ we have $x - a = 0$ or $x - b = 0$.

In other situations we might need to combine conditions with a logical *and* operation instead, using

$$p(x) = 0 \text{ and } q(x) = 0 \iff |p(x)| + |q(x)| = 0.$$

The use of the absolute value will be key in what follows. However, we have to decide what functions are permitted in forming an equation. Euler defined 'function' in volume 1 of his *Introduction to the analysis of the infinite*:

> *A function of a variable quantity is an analytic expression composed in any way whatsoever of the variable quantity and numbers or constant quantities.* Hence every analytic expression, in which all component quantities except the variable z are constants, will be a function of that z; thus $a + 3z$, $az - 4z^2$, $az + \sqrt{a^2 - z^2}$, c^z, etc are functions of z.

Here, Euler proposes that a *function* is that which can be expressed using an analytic expression. So what variables and operations can be used to make up an *analytic expression*?

An analytic expression is anything that can be built from a short list of operations. First, we can involve constants, via addition, subtraction, multiplication or division. Next, a few 'basic' functions. These are usually powers and roots of z, exponentials, logarithms, and trigonometric functions.

The absolute value function is sometimes defined as $|x| = \sqrt{x^2}$, and in the example above Euler makes use of square roots, so we can claim it as an analytic expression.

A technical side note is that the use of the absolute value function *does* allow zero to be factored over the collection of arbitrary algebraic expressions:

$$0 = (x + |x|)(x - |x|)$$

is the difference of two squares. If we use the absolute value we have serious theoretical problems because we introduce zero divisors. Furthermore, practical algebraic manipulation of expressions with the absolute value can be difficult. Despite these practical problems, the absolute value function is included in our collection of legitimate functions to use!

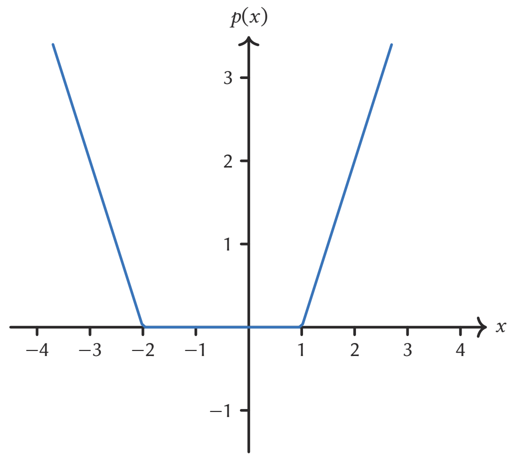
To find the equation of a triangle we will therefore make use of implicit functions (to create an equation), complex numbers (to abbreviate and clarify) and synthesis (to separate conditions). We now have our complete set of ingredients.

The equation of a triangle

Finally we address the problem: given three points A, B and C in the plane, represented by complex numbers a, b and c respectively, find an equation for the triangle with these points as vertices.

Our first task is to find the equation of a single segment. We start with an interesting piecewise real function. Here's the graph of

$$p(x) := |x - 1| + |x + 2| - 3.$$



Note that the set of real solutions $p(x) = 0$ is precisely the real interval $[-2, 1]$. When we add together two shifted absolute value functions, namely $|x - 1|$ and $|x + 2|$, there is a segment on which the sum is constant. It's then just a simple matter to shift this vertically to build the function where the set of real solutions $p(x) = 0$ is precisely a real interval.

When $a, b, z \in \mathbb{C}$ both $|z - a|$ and $|z - b|$ remain real quantities, so we can use exactly the same idea in the complex plane. More generally we assume that $a, b \in \mathbb{C}$ and define

$$p(z) := |z - a| + |z - b| - |a - b|\,.$$

Then we will show, a little formally perhaps, that $p(z) = 0$ if and only if $z \in \mathbb{C}$ lies on the segment joining a and b, including the end points.

In a moment we will need the following, with a new variable t. Note that for $z \neq 0$, $|tz| = t|z|$ if and only if t is a non-negative real number.

Further note that $z \in \mathbb{C}$ lies on the segment joining a and b, including the end points, if and only if $z = ta + (1 - t)b$ for $t \in [0, 1]$. This parametric equation, with parameter t, is a very useful function.

Assuming $a \neq b$ and $z \in \mathbb{C}$, defining t as $t = (z - b)/(a - b)$, and rearranging for z, we have

$$z = ta + (1 - t)b.$$

This means that we can rewrite $p(z)$ in the form

$$\begin{aligned} p(z) &= |ta + (1-t)b - a| \\ &\qquad + |ta + (1-t)b - b| \\ &\qquad - |a-b| \\ &= |(1-t)(b-a)| + |t(a-b)| - |a-b| \\ &= (1-t)\,|b-a| + t\,|a-b| - |a-b| \quad \text{for } t \in [0,1] \\ &= (1-t)\,|a-b| + t\,|a-b| - |a-b| \\ &= 0. \end{aligned}$$

Hence $p(z) = 0$ if and only if both t and $1-t$ are positive–that is, for $t \in [0,1]$.

In the case in which $a = b$, $p(z) = 2\,|z-a| = 0$ if and only if $z = a = b$, ie if $z \in \mathbb{C}$ lies on the (trivial) segment joining a and b, including the end points.

Now we have the equation of a single segment, we can combine these segments as a product. Let $a_1, \dots, a_n \in \mathbb{C}$ be given, set $a_{n+1} := a_1$ and define

$$P(z) := \prod_{k=1}^{n} \big(|z-a_k| + |z-a_{k+1}| - |a_k - a_{k+1}|\big),$$

then $P(z) = 0$ if and only if z lies on the polygon with $a_1, \dots, a_n$ as vertices.

Really to show this we note that $P(z) = 0$ if and only if at least one of the factors of $P(z)$ is zero. Each of these factors is in the form given opposite, so $P(z) = 0$ if and only if z lies on one of the segments joining a_k to a_{k+1} (for $k = 1, \dots, n$).

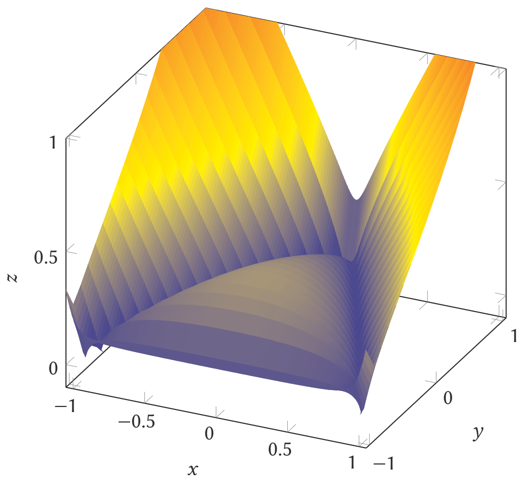
For example, let

$$a_1 = \mathrm{i}, \;\; a_2 = -1-\mathrm{i}, \;\; a_3 = 1-\mathrm{i},$$

then the triangle with these vertices is the set of $z \in \mathbb{C}$ for which

$$\begin{aligned} P(z) &= \big(\,|z+\mathrm{i}-1| + |z-\mathrm{i}| - \sqrt{5}\,\big) \\ &\quad \times \big(\,|z+\mathrm{i}+1| + |z-\mathrm{i}| - \sqrt{5}\,\big) \\ &\quad \times \big(\,|z+\mathrm{i}+1| + |z+\mathrm{i}-1| - 2\,\big) = 0. \end{aligned}$$

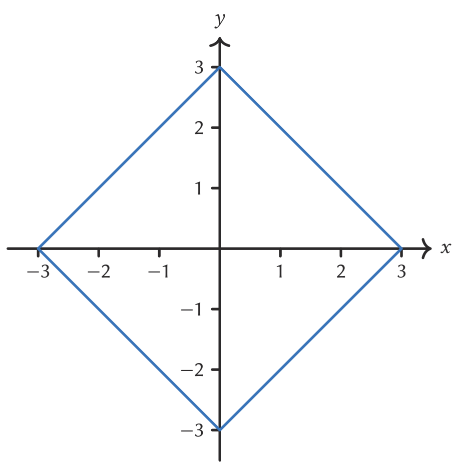
Here's $|P(z)|^{1/3}$ for this example:



We add the 1/3 power here to make the behaviour near $|z| = 0$ look more like $|z|$ and less like $|z|^3$. This makes rootfinding more stable numerically and makes the three dimensional plot more reasonable to see.

The downside of the approach of the product formula is in ‘simplifying’ the resulting product. For example, it is relatively simple to see that a square with vertices at ± 3 and $\pm 3\mathrm{i}$ has equation

$$|x| + |y| = 3.$$



Applying the product as seen above, we have the square as

$$\begin{aligned}&(|z-3|+|z-3\mathrm{i}|-|3-3\mathrm{i}|)\\ &\times(|z-3\mathrm{i}|+|z+3|-|3+3\mathrm{i}|)\\ &\times(|z+3|+|z+3\mathrm{i}|-|3-3\mathrm{i}|)\\ &\times(|z+3\mathrm{i}|+|z-3|-|3+3\mathrm{i}|).\end{aligned}$$

It is not at all clear how, with algebraic re-writing rules, to substitute $z = x+\mathrm{i}y$ and then transform the complex product into $|x|+|y| = 3$. Showing that would be an interesting challenge! Hence, the product notion is likely to be theoretically useful rather than practical in computations.

Closing

In order to build our 'equation of a triangle' we made use of complex numbers and a range of functions, including the absolute value function. We made use of implicit functions (to create an equation), complex numbers (to abbreviate and clarify) and synthesis (to separate conditions). However, the use of the absolute value function, and the way in which we used it, mean the resulting equations are probably difficult in practice to manipulate algebraically into alternative equivalent forms. This formula for the equation of a triangle is therefore probably more interesting from a theoretical than practical perspective.

There are plenty of other situations in which a function with particular properties is created by synthesis. For example, in 1964, CP Willans from the University of Birmingham gave

$$p_n = 1+\sum_{m=1}^{2^n}\left[\sqrt[n]{n}\left(\sum_{x=1}^{m}\left[\cos^2\pi\frac{(x-1)!+1}{x}\right]\right)^{-1/n}\right]$$

as a formula for the nth prime number. An explanation of the synthesis of this formula was recently given by Eric Rowland on his YouTube channel. This formula uses relatively mainstream functions, but the nth prime requires calculations up to 2^n which is catastrophically inefficient for even modest primes.

More recently, in 1975, James Jones from the University of Calgary suggested

$$p_n = \sum_{i=0}^{n^2}\left(1 \mathbin{\dot{-}} \left[\left(\sum_{j=0}^{i}\left[(j \mathbin{\dot{-}} 1)!^2 \bmod j\right]\right) \mathbin{\dot{-}} n\right]\right),$$

in which the nth prime requires calculations up to n^2, which is considerably better than 2^n. The symbol $\dot{-}$ is 'proper subtraction', defined as

$$x \mathbin{\dot{-}} y := \begin{cases} x-y & \text{if } x \geqslant y,\\ 0 & \text{if } x < y.\end{cases}$$

Are these functions legitimate, or do they simply hide the hard work elsewhere?

There are plenty of examples of formulae $p(n)$ which give the nth prime number.

None of these formulae for the nth prime are useful in any computational situation because they are all too inefficient to calculate. They are theoretically interesting, however, and they form another collection of functions which look very complex on first acquaintance but which have been synthesised carefully to have particular properties, just as we have done here for the equation of the triangle.

Chris Sangwin

Chris joined the University of Edinburgh in 2015 as professor of technology enhanced science education. His learning and teaching interests include digital educational technology and automatic assessment of mathematics using computer algebra.

✉ c.j.sangwin@ed.ac.uk

THE BIG ARGUMENT

Should we write another issue?

YES!

argues SAM KAY

The reason we spend months writing this magazine is because the fans love it. It's cool, it's hip, it's *Cosmo*. Don't believe me? Here's a proof by induction.

It's trivial that the first ever issue of *Chalkdust* was fun to read. If not for its great initial success, I wouldn't be here writing!

Our ever-expanding mailbox and letters pages tells us it is safe to assume that n issues of *Chalkdust* are fun to read.

Now, take a group of $n+1$ issues of *Chalkdust* and consider only the first n issues. From the above assumption, these n issues are fun to read. Likewise, consider the last n issues of *Chalkdust*. These issues also have to be fun to read by the same assumption!

Since both the first n issues and the last n issues overlap by at least one issue, this overlapping issue will be as fun to read as the rest, and thus all $n+1$ issues of *Chalkdust* are fun to read.

If n issues of *Chalkdust* are fun to read, then we have proven that $n+1$ issues of *Chalkdust* are also fun to read. And we already noted that the first issue was indeed fun to read. By induction, any issue of *Chalkdust* we release will be fun to read. So we probably should release another.

NO!

argues THE REST OF THE TEAM

Have you seen the magazine?!

Sure, we have a lovely selection of articles by lovely authors, and you lot are a lovely audience... but have you seen the pages between the articles? It's all low effort trash, placeholder text we forgot to update, inconsistent use of Oxford commas, and repeating the word 'lovely' multiple times in the same sentence.

Surely by now, a decent publication would have accidentally come up with at least one good joke.

Half the reviews are of one-off events that you've missed.

The letters that Dirichlet receives aren't even real.

We've been on *Have I Got News For You*—the ultimate aim of any niche publication—but where else is there to go? In the bin.

We realised early on that they'll let you print anything, but should they really be printing this?!

Then again, printing this is quite fun, and we already have one passable pun for Dear Dirichlet 21. Better write the rest of the magazine then.

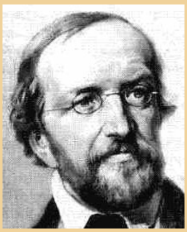
dear DIRICHLET

Moonlighting agony uncle Professor Dirichlet answers your personal problems. Want the prof's help? Contact ✉ deardirichlet@chalkdustmagazine.com

Dear Dirichlet,

I'm the groundskeeper at Edgbaston cricket ground, and England are playing a test match next month. The captain has not-so-subtly suggested to me that before England go out to bat, I should cut the grass extremely short under the rope around the outside. Is this friendly gardening advice, or does he have something up his sleeve?

– Mow money, mow problems, Selly Oak



■ **DIRICHLET SAYS:** I've heard of your captain's technique before. To reduce spin on the field, just remove the field at the boundary. A simple consequence of

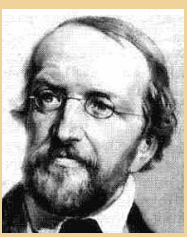
$$\iint_S (\underline{\nabla} \times \underline{F}) \cdot d\,\underline{S} = \oint_{\partial S} \underline{F} \cdot d\,\underline{r},$$

or as the pundits call it, Stokes' theorem. Of course, I personally recommend zero as your boundary condition.

Dear Dirichlet,

I'm supposed to have been writing a dissertation about triangles, but I've spent most of the year slacking off instead. I had a great idea: my housemate's also working on a triangle project, and he's been writing, rewriting, and binning chapters at a record pace. I took a bunch of pages he wasn't using, stuck them together, increased the font size, and submitted it—but I got a 100% plagiarism score on TurnItIn! Now what?!

– Orla Coppie, London SE15



■ **DIRICHLET SAYS:** Ah, this is a classic 'ship of thesis-us' situation (also known as Trig-ger's broom). Each time your housemate cut a chapter and replaced it with something better, you got pieces of his work, but the final dissertation was still his. Scaling it up won't help - the triangles will still be flagged as similar! My advice: use $\hat{c}$GPT.

Dear Dirichlet,

Wow. That opening Olympics ceremony! The incredible weather. The Assassins Creed horse thing. And the power of that Céline Dion performance. I can't believe I've never heard of her before! Where should I start exploring her catalogue?

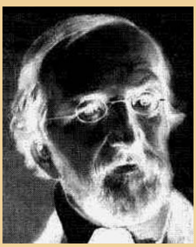
– Rose Water, Newcastle-under-Whelming

■ **DIRICHLET SAYS:** When I'm covering Céline on karaoke, I start with 'All summing fractal me now', followed with 'All bi(na)ry self', and, bien sûr, 'Pie chart will go on'. For the more discerning, I do a great rendition of 'Where does Descartes eat now?' (he's banned from Burger King), and I always end the night with the undisputed Eurovision banger, 'Poincaré pas sans moi'.

Dear Dirichlet,

I'm teaching a large class of undergraduates next term, and I'm worried my voice isn't going to carry. I've got a number of small Bluetooth speakers and I think I read that they work better if set at better relative frequencies. I've done some simulations with $\sqrt{2}$ and $\sqrt{3}$, but the results were underwhelming. Where am I going wrong?

– Sonia Sonic, The Wirral



■ **DIRICHLET SAYS:** The time for theory has passed. You have to get in the lecture theatre and do it for real. Bring all your speakers, and set their frequencies to a half, a quarter, an eighth,... You'll find this is much more effective since fractions peak louder than surds.

Dear Dirichlet,

I'm really confused by my first year analysis course. Can you help?

– Epsilon help-ta, Carrickfergus

■ **DIRICHLET SAYS:** For every undergraduate, there is a lecture number N such that for all n > N, the undergraduate is lost.

ON THE COVER

Spirographs

Ashleigh Wilcox, Jenny Power and Rachel Evans
show you how to draw pretty pictures

SPIROGRAPHS blur the line between geometry and art. When spirographs are mentioned, a first thought probably goes to the kids' art toy—but who says it's just for kids?! After this issue's cover artist introduced me to spirographing at a Piscopia Initiative event, my first task was a trip to Smyths toys superstores to grab myself a kit. The shapes of a spirograph (hypotrochoids and epitrochoids—but we will get to that later) are created by using two circles. One circle is stationary; we will call this a ring because it has a large hole in the centre. Around the inside and outside of the ring are small teeth, or notches. The other circle we use moves and we will call it a cog. The cog is a smaller filled circle with small holes for your pen, and teeth around the outside. These teeth help with the rotations of the circle and give guidelines for where to start. In short, you take a ring, put your cog on the inside (or outside), line up the teeth, put your pen in the hole and go around and around the circle. Although it sounds simple, the creation of spirographs takes practice and is a slow art form.

About the artist

One person who got this kit as a child and never gave it up is Rachel Evans. She is better known in the community as Spirograph Girl. When I asked her about her inspirations for choosing spirographs as an art form, she said this: "I would say that as an artist I found spirograph but that's completely untrue! Spirograph made me into an artist and it's been an amazing ride, just going through one project after another. My process is very simple: I spirograph, colour in the pattern, then cut it out and stick it on. I also work digitally via the Inspiral app—which is what I used for this cover, I hope you like it!"

Spirograph Girl

If you like Rachel's artwork, why not take a minute to look at her website or follow her on social media and let her know!

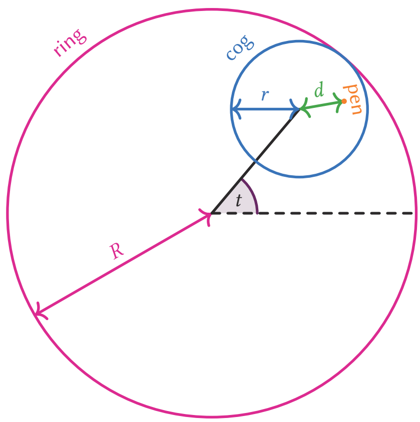
Spirograph Girl
@spirographgirl
spirographgirl.com
spirographgirl@gmail.com

Hypotrochoids

Let's get into the maths! Most of the spirographs on the cover are hypotrochoids. These are roulette curves which are created when a circle rolls around the inside of a fixed circle. Consider the diagram of a typical spirograph setup.

We have the following variables:

- R: the radius of the fixed ring.
- r: the radius of the rotating cog.
- d: the distance between the pen and the centre of the cog.



↑ Schematic of spirograph setup

The equations of the two circles are given by

$$\begin{aligned}\text{Ring:}&\quad x^2+y^2=R^2\\ \text{Cog:}&\quad (x-(R-r))^2+y^2=r^2.\end{aligned}$$

By examining the path the pen takes as the cog rotates around the ring, we get the following pair of parametric equations for the hypotrochoid:

$$\begin{aligned}x(t)&=(R-r)\cos(t)+d\cos\left(\frac{R-r}{r}t\right)\\ y(t)&=(R-r)\sin(t)-d\sin\left(\frac{R-r}{r}t\right).\end{aligned}$$

Note here that the parametric variable t is not the polar angle, but the angle between the centre of the rotating cog and the horizontal axis. To complete the full pattern, this angle t takes values from 0 (when the pattern starts) to $2\pi \times \mathrm{LCM}(r,R)/R$ (when the pattern is complete). Here, LCM denotes the *lowest common multiple* between the radii of the cog and ring. Interestingly, we can use this to determine the number of spokes a spirograph pattern will have, and how many rotations of the cog it takes for the pattern to be complete! We have

$$\text{number of rotations}=\frac{\mathrm{LCM}(r,R)}{R},$$

and

$$\text{number of spokes}=\frac{\mathrm{LCM}(r,R)}{r}.$$

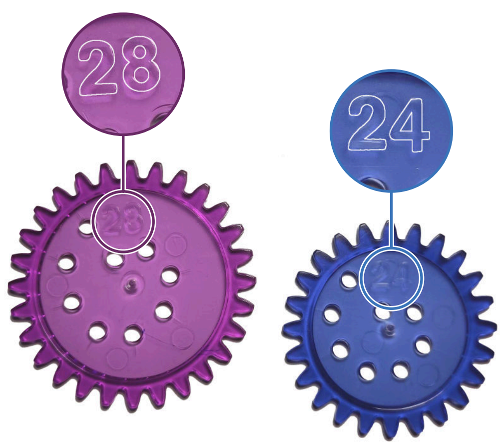
What we can notice is that only the radii of the two circles affect the number of spokes in your pattern. The pen location d has no impact on this. The pen location only changes how 'spiky' your pattern is. When your pen is close to the centre of the cog, you will obtain very wide and short spokes, and when the pen is closer to the edge of the cog, you will get more tall and narrow spokes.

In particular it is the *ratio* between the two radii that is most important. Let's call this ratio

$$k=\frac{R}{r}.$$

We get different behaviour depending on the value of k. If k is a rational number, given in its simplest form as $k=a/b$, the number of spokes is given by a and the number of rotations are given by b. This is equivalent to the LCM formulation presented above. We get some interesting examples when k is an integer (but we will look at those later). If k is irrational the pattern is infinite and will never complete.

Now, before you all take out your rulers and attempt to measure the radii of your spirograph parts, there is an easier way to calculate these. Each spirograph part comes labelled with the



↑ The cogs and rings of a spirograph usually have the number of teeth written on them number of teeth it has (the rings have two numbers for the inner and outer circles). We can rewrite the formulae in terms of the number of teeth on the rings and cogs instead. Let's say that N represents the number of teeth on the ring and n represents the number of teeth on the cog. If the distance between the teeth is uniform across all parts, and this distance is, say, S, then the circumferences of the wheels are given by

$$L = NS, \qquad \ell = nS.$$

However, we know that the circumference of the circle is given by

$$L = 2\pi R, \qquad \ell = 2\pi r.$$

From simple manipulation, we obtain

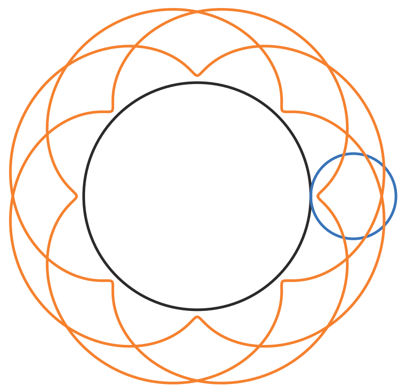
$$R = \frac{S}{2\pi}N, \qquad r = \frac{S}{2\pi}n,$$

and end up with the much easier to work with formulae

$$\text{number of rotations} = \frac{\mathrm{LCM}(n, N)}{N},$$
$$\text{number of spokes} = \frac{\mathrm{LCM}(n, N)}{n}.$$

Epitrochoids

Another type of roulette curve created by a spirograph is an *epitrochoid*. This curve is created when a circle rolls around the outside of a fixed circle. Following the path the pen takes as the cog rotates around the outside of the ring (as we did for hypotrochoids), we get the following pair of parametric equations for the epitrochoid:



↑ Epitrochoid with $R = 8$, $r = 3$ and $d = 2.5$

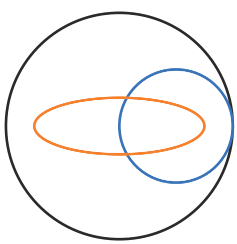
$$x(t) = (R + r)\cos(t) - d\cos\left(\frac{R+r}{r}t\right)$$
$$y(t) = (R + r)\sin(t) - d\sin\left(\frac{R+r}{r}t\right).$$

Ellipses and rounded polygons

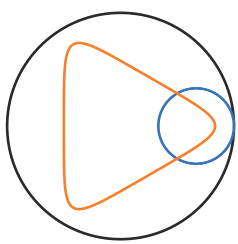
The first example we'll look at is when the ratio of the ring to the cog is an integer, $R = kr$. The parametric equations are given by

$$x(t) = (k-1)r\cos(t) + d\cos((k-1)t),$$
$$y(t) = (k-1)r\sin(t) - d\sin((k-1)t).$$

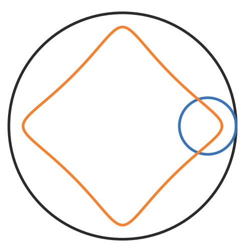
↓ Ellipse and rounded polygons with $d = r/2$



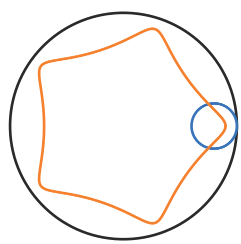
↑ $k = 2$ (ellipse)



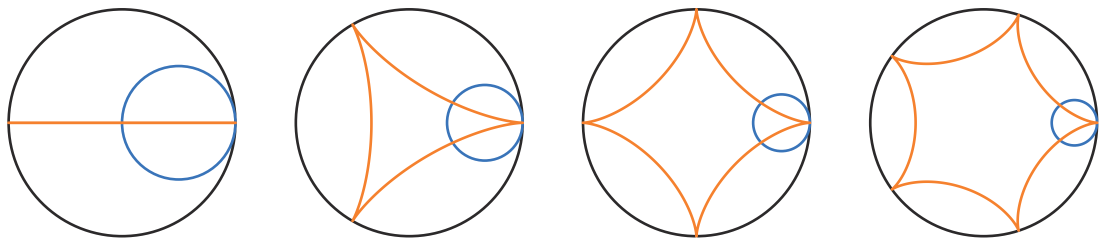
↑ $k = 3$



↑ $k = 4$



↑ $k = 5$



↑ $k = 2$ (Tusi couple) ↑ $k = 3$ (destoid) ↑ $k = 4$ (astroid) ↑ $k = 5$ (pentoid)

↑ Hypocycloids with $R = kr$ and $d = r$

Using our formulae on the previous page, the number of spokes the pattern will have is the integer k and it will only take one rotation to complete the pattern! Additionally, if k is even the pattern will be symmetric about the y-axis. A very special case is when $k = 2$, as this produces an ellipse!

If you have a standard spirograph kit at home, you can see this behaviour with the 144/96 ring and the 48, 32, 24 cogs ($k = 2, 3, 4$ in these cases). Note that this only works if $d \neq 0$, ie your pen is not in at the centre of the cog. Regardless of the choice of k, for $d = 0$ you will always get a circle!

Hypocycloids

The second example we'll look at is when the pen is on the circumference of the cog ($d = r$). This is not something we can draw with a physical spirograph kit unfortunately, but the resulting shapes are still very cool to look at. The equations for these are the exact same as for a hypotrochoid except with $d = r$. The spokes obtained in hypocycloids are what are known as *cusps*, sharp corners where the curve is not differentiable.

A famous example of a hypocycloid is when $R = 2r$. This results in a 2-cusped hypocycloid which is just a line segment! This is often referred as a 'Tusi couple' named after Persian mathematician and astronomer Nasir al-Din al-Tusi (1201–1274):

$$x(t) = (R-r)\cos(t) + r\cos\left(\frac{R-r}{r}t\right),$$
$$y(t) = (R-r)\sin(t) - r\sin\left(\frac{R-r}{r}t\right).$$

Rosettes

Last, but certainly not least, we have my favourites—the rosettes. This example produces spirograph patterns that look like flowers. This occurs when we choose $d = R - r$. The parametrised equations for this curve are given by

$$x(t) = (R-r)\left[\cos(t) + \cos\left(\frac{R-r}{r}t\right)\right],$$
$$y(t) = (R-r)\left[\sin(t) - \sin\left(\frac{R-r}{r}t\right)\right].$$

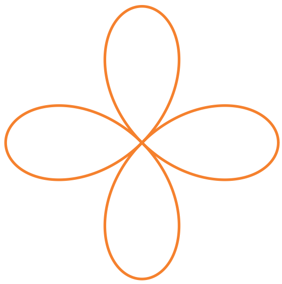
However, for rosettes it is much more convenient to write the equations in polar form instead. After doing some clever trigonometry, we end up with the polar equation

$$\rho = 2(R-r)\cos\left(\frac{R}{R-2r}\theta\right),$$

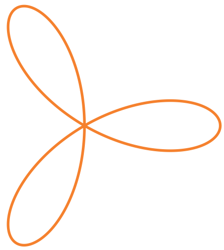
where ρ is the distance from the curve to the origin and θ is the polar angle. In this formulation, it is the ratio

$$p = \left|\frac{R}{R-2r}\right| = \left|\frac{k}{k-2}\right|$$

that determines how many spokes (or 'petals') our flower will have. If p is an integer we get a special pattern called a *rose curve* as it resembles a petalled flower. If p is odd the flower will have p petals and if p is even the flower will have $2p$ petals. If p is not an integer, you will get a rosette with the number of spokes given by the formulae introduced earlier for hypotrochoids.



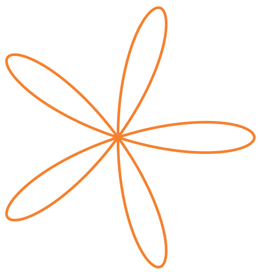
↑ Rose curve, $p = 2$



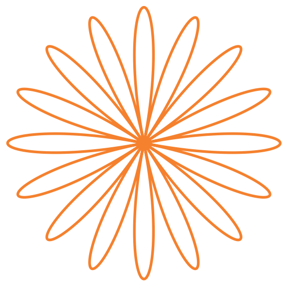
↑ Rose curve, $p = 3$

Rose curves & rosettes

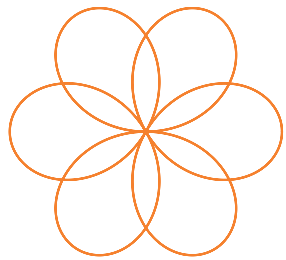
If you own a spirograph kit, try to make a $p = 4$ rose curve using the 144/96 wheel and the 60 cog, placing your pen in the 7th hole from the outside. Send your attempts to *Chalkdust* and the prettiest ones will feature on our socials.



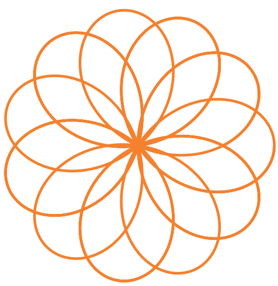
↑ Rose curve, $p = 5$



↑ Rose curve, $p = 8$

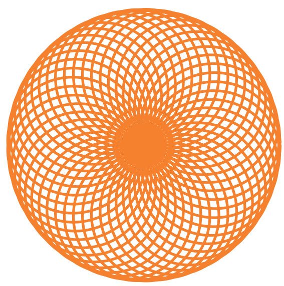


↑ $p = 3/2$



↑ $p = 11/7$

↑ $p = 23/9$



↑ $p = 49/45$

Ashleigh Wilcox

Ashleigh is a PhD student and graduate teaching assistant at the University of Leicester. Her main mathematical interests are in number theory. She is passionate about outreach and inclusion in mathematics, volunteers as a Stem ambassador and is a co-lead of the Piscopia Initiative.

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Jenny Power

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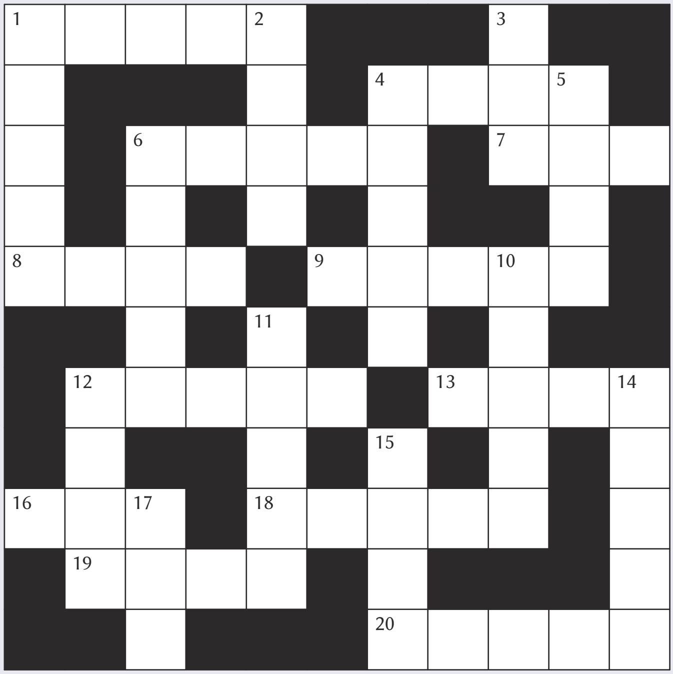
PUZZLES

Looking for a fun puzzle but not got time to tackle the crossnumber? You're on the right page.

Take two

Remove two digits from each of the clues then write the answers in the crossword grid. As usual, no entries begin with 0.

Across		Down	
1	1872605	**1**	3782470
4	765432	**2**	605316
6	2321242	**3**	14306
7	18331	**4**	2614985
8	407480	**5**	123456
9	3803454	**6**	5032801
12	5301268	**10**	5039082
13	909105	**11**	9628031
16	10602	**12**	356809
18	3087982	**14**	9015109
19	238573	**15**	590905
20	1999991	**17**	60253



Mathematical songs

Guess the song titles from the mathematical clues. The artists are given in brackets under each song.

$$x \text{ such that } |x-u|<\varepsilon$$
(The Carpenters)

$$\mathbb{P}(\text{I will die}) = 0$$
(Gloria Gaynor)

$$\{\text{me}\} \cup \{\text{me}\} \cup \{\sqrt{-1}\}$$
(De La Soul)

$$\sum_{\substack{1 \leqslant i < \text{day} \\ i \mid \text{day}}} i = \text{day}$$
(Lou Reed)

$$\tan(\times)\cos(\times)$$
(Prince)

$$\text{hot} \notin \{x : x \text{ to stay}\}$$
(Chappell Roan)

$$(\text{fire}, +) \text{ is an Abelian group}$$
$$(\text{fire}, \cdot) \text{ is a monoid}$$
$$\forall a, b, c \in \text{fire}, a\cdot(b+c) = a\cdot b + a\cdot c$$
$$\forall a, b, c \in \text{fire}, (a+b)\cdot c = a\cdot c + b\cdot c$$
(Johnny Cash)

Cor! Swords!

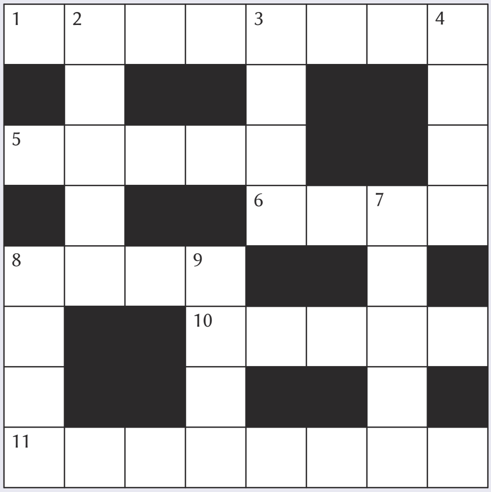
Solve the anagrams and write the answers in the crossword grid.

Across

1 Alerting
5 Rule e
6 Demo
8 Dees
10 I am eg
11 Tried sec

Down

2 To rue
3 Morn
4 Geed
7 I dig t
8 Ruds
9 Scid



L M	O E	M D	O E	L N
E M		O R		O E
S A	X T	I A	R O	T M
N M		N O		M T
G S	I A	M G	E M	S A

Cross out

Cross out one letter in each square so that three five-letter words are spelt out in each direction.

Arrange the digits

Put the numbers 1 to 9 (using each number exactly once) in the boxes so that the sums are correct. The sums should be read left to right and top to bottom ignoring the usual order of operations. For example, $4 + 3 \times 2$ is 14, not 10.

$$
\begin{array}{ccccccc}
\square & \div & \square & \times & \square & = & 3 \\
\times & & \div & & - & & \\
\square & \div & \square & \times & \square & = & 35 \\
- & & - & & \times & & \\
\square & \div & \square & \times & \square & = & 24 \\
= & & = & & = & & \\
6 & & 1 & & 9 & &
\end{array}
\qquad
\begin{array}{ccccccc}
\square & \div & \square & \times & \square & = & 4 \\
+ & & - & & + & & \\
\square & \times & \square & \times & \square & = & 15 \\
- & & \times & & - & & \\
\square & - & \square & \times & \square & = & 9 \\
= & & = & & = & & \\
0 & & 18 & & 2 & &
\end{array}
$$

Poly-π

Clem Padin calculates π, but not as we know it

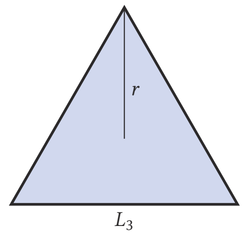
WHILE reading William Dunham's *The Mathematical Universe*, I came across this:

> *A novice, introduced to circles for the first time, would rather quickly recognise an essential fact: All circles have the same shape... We note by contrast that not all triangles have the same shape, nor all rectangles. Behind this unexciting observation lies one of the profound theorems in mathematics: that the ratio of the circumference to the diameter is the same for one circle as for any other.*

What does Dunham mean by "all circles have the same shape"? We would all agree that this is true, regardless of a circle's size, but more formally, we can say all circles are geometrically similar. And while this is obviously true for circles, it is equally true for equilateral triangles or squares. And it occurred to me that this "profound theorem" Dunham refers to is as true for these polygons as it is for the circle, provided we can define their diameters. This is easy for a certain subset of polygons: regular polygons! Just as π represents the constant ratio of circumference to diameter for all circles of all sizes, regular polygons have an analogous constant, which I've named 'polygon-π', or 'poly-π' for short.

The triangle

Let's start with the equilateral triangle. For consistency, I'll define the triangle's radius r as the distance from the centre of the triangle to any vertex. Its diameter is then twice its radius.

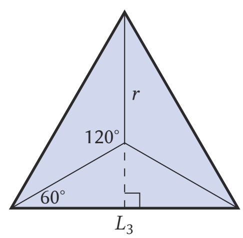


For an equilateral triangle with sides of length L_3, the counterpart to the circle's circumference is, of course, the triangle's perimeter, $3L_3$. So, we can define the π_3 value for all equilateral triangles as

$$\pi_3 = \frac{3L_3}{2r},$$

where the subscript '3' indicates the number of sides.

To find L_3 in terms of r, we can draw a line from the centre of the triangle to each vertex. This divides the equilateral triangle into three equal isosceles triangles with sides of length r and three central angles of 120°.



Dropping a perpendicular from the centre to any side divides L_3 and the central angle in half, creating a right-angled triangle and allowing us to observe the following relation:

$$\sin(60^\circ) = \frac{L_3}{2r}.$$

Replacing $L_3/2r$ with $\sin 60^\circ$ in the expression for

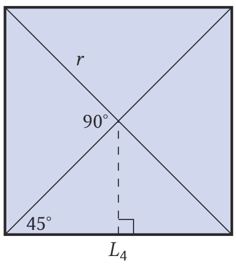
π_3 gives us

$$\pi_3 = 3\sin(60°) \approx 2.598.$$

This value is the equilateral triangle's poly-π value! So, for any equilateral triangle, the ratio of the triangle's perimeter to its 'diameter' equals π_3 or about 2.6.

The square

Let's now examine the square. Again, we define a radius r as the length from the square's centre to a vertex and the length of one side as L_4.



Now the ratio of the square's 'circumference' to its 'diameter' can be written as

$$\pi_4 = \frac{4L_4}{2r}.$$

Note that the central angles in the square are 90°and the four radii divide the square into four isosceles triangles with base angles of 45°.

Again, dropping a perpendicular divides L and a central angle so we can write

$$\sin(45°) = \frac{L_4}{2r}.$$

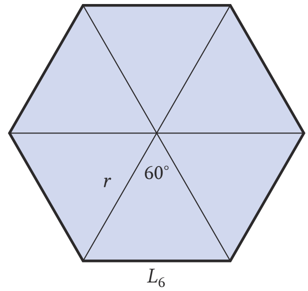
Substituting $\sin(45°)$ for $L_4/2r$ in our expression for π_4, we get

$$\pi_4 = 4\sin(45°) \approx 2.828.$$

This is the square's poly-π value.

The hexagon

The poly-π value for a hexagon turns out to be quite simple.



Note that the inner angles are $360°/6 = 60°$. The other angles in the six triangles of the hexagon are then also 60° and so we have six equilateral triangles. This then means that $r = L_6$. As a result,

$$\pi_6 = \frac{6L_6}{2r} = \frac{6L_6}{2L_6},$$

and so

$$\pi_6 = 3.$$

n-sided polygons

In each subsequent regular polygon we employ the same technique. That is, the nth poly-π is given by

$$\pi_n = \frac{nL_n}{2r} = \frac{P_n}{2r},$$

where P_n is the perimeter.

A perpendicular drawn from the central angle $360°/n$ divides L and the central angle such that

$$\sin\left(\frac{1}{2} \times \frac{360°}{n}\right) = \frac{L_n}{2r},$$

so that

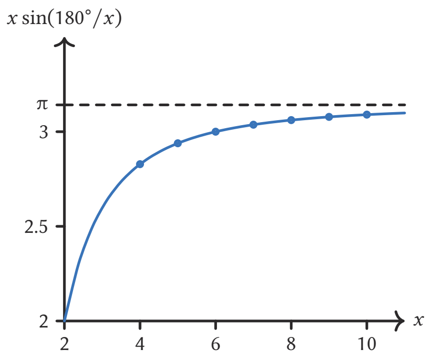
$$\pi_n = n\sin\left(\frac{180°}{n}\right).$$

This now is the poly-π value for a regular polygon with n sides. Note how the result does not depend on any physical characteristics of the polygon except the number of sides!

The table overleaf displays the poly-π for several polygons. The values have been rounded.

number of sides	poly-π
3	2.608
4	2.828
5	2.939
6	3.000
7	3.037
8	3.061
9	3.078
10	3.090
100	3.14108
1000	3.141587

And here's a plot of the poly-πs up to 10:



We can see (as we would expect) that as $n \to \infty$, $\pi_n \to \pi = 3.14159\ldots$

Using poly-π

When given the radius of a circle, we can calculate the circumference, and vice versa, because of the constant of proportionality between them expressed as π. This is also the case for a regular polygon because of the existence of its poly-π. So for a pentagon whose perimeter is, say, 15 we can find its radius (the distance from the centre of the pentagon to a vertex) by rearranging equation () to get

$$r = \frac{P_5}{2\pi_5}.$$

Using the table above we find that $\pi_5 = 2.94$, so

$$r = \frac{15}{2(2.94)} \approx 2.55$$

Unlike a circle, however, in addition to using poly-πs to calculate the perimeter or radius, we can also calculate the length of a polygon's side. Using equation () and the value derived above for the radius, we get

$$L_5 = \frac{2\pi_5 r}{5} \approx \frac{2 \times 2.94 \times 2.55}{5} \approx 3.0,$$

which is, of course, what we should get as the perimeter is 15 and the figure is a pentagon, so each side is $15/5 = 3$.

Poly-π and area

The area of each n-sided polygon is given by the sum of the areas of the n inscribed triangles. A perpendicular drawn from the central angle to a side (L_n, the triangle's base) gives us the height of one of the inscribed triangles and can be written as

$$r\cos\left(\frac{1}{2} \times \frac{360^\circ}{n}\right) = r\cos\left(\frac{180^\circ}{n}\right).$$

So, for a regular n-sided polygon, its area is given by

$$A_n = n\left(\frac{1}{2}L_n\right)\left[r\cos\left(\frac{180^\circ}{n}\right)\right] = \frac{nrL_n}{2}\cos\left(\frac{180^\circ}{n}\right).$$

Solving equation () for L_n we get

$$L_n = \frac{2\pi_n r}{n}.$$

Combining these last two equations, we can write the area of an n-sided polygon in terms of r, its radius, as

$$A_n = \pi_n r^2 \cos\left(\frac{180^\circ}{n}\right). \qquad ()$$

Note how similar this equation is to the equation for the area of a circle! And, in fact, note that as $n \to \infty$, $180^\circ/n \to 0$, and $\cos(180^\circ/n) \to 1$. Thus, we can state that for n-sided polygons, as the number of sides increases or as $n \to \infty$, $A_n \to \pi r^2$.

We can also express the area in terms of the length of a side. Using equation () but this time solving for r, we get

$$r = \frac{nL_n}{2\pi_n}.$$

Substituting this into equation (☙❧), we get

$$A_n = \frac{1}{\pi_n}\left(\frac{nL_n}{2}\right)^2 \cos\left(\frac{180^\circ}{n}\right). \qquad (❧❧❧)$$

Though not as elegant as equation (☙❧), it does allow us to determine the area of a regular polygon given the length of one side. Again we can consider what happens as $n \to \infty$. If we have a fixed side length L, our area formula becomes

$$A_n = \frac{1}{\pi_n} n^2 \left(\frac{L}{2}\right)^2 \cos\left(\frac{180^\circ}{n}\right).$$

Now, we know $1/\pi_n \to 1/\pi$, and $\cos(180^\circ/n) \to 1$ as before. Since we're holding the side length constant, $(L/2)^2$ is just a constant. But n^2 certainly goes to infinity! In hindsight, this is obvious—if we keep the same side length, but add more and more sides then clearly the area must get larger and larger.

Using poly-π to calculate the area of a regular polygon

Using equation (☙❧) we can easily find the area of any regular polygon if we know its radius. So for a hexagon with a radius of 12, we can find its area as follows:

$$A_6 = \pi_6 r^2 \cos\left(\frac{180^\circ}{6}\right).$$

We know π_6 is exactly 3, so

$$A_6 = (3)(12)^2 \cos\left(\frac{180^\circ}{6}\right) = 432\cos(30^\circ).$$

What is the length of a hexagon's side given such an area? Solving equation (❧❧❧) for L_n, we find

$$L_n = \frac{2}{n}\left(\sqrt{\frac{A_n \pi_n}{\cos(180^\circ/n)}}\right).$$

For the above hexagon, that is

$$\begin{aligned} L_6 &= \frac{2}{6}\sqrt{432 \times 3 \times \cos(30^\circ)/\cos(30^\circ)} \\ &= \frac{1}{3}\sqrt{1296} = 12. \end{aligned}$$

Although, we knew that from the start—given this is a hexagon, we already know that the radius and the length of a side are equal. Since we proposed a radius of length 12, the side (L_6) should also be of length 12.

Conclusion

A π-like value exists for regular polygons. It emerges naturally once we noticed that regular polygons at different scales have 'the same shape', just like circles. Given a measurement for a polygon, poly-πs make it easy to find others. For example, from the side length we can find the distance from its centre to any vertex (its radius); from the area we can find the perimeter.

Additionally, this concept might help people get a better idea of what π really means. The special symbol and the 'funny' name of π may create a mystique (not entirely undeserved) that can distract from its true meaning. Introducing poly-πs helps to clarify that π is not a mysterious quantity exclusive to circles but rather a simple ratio—and such ratios appear in many different geometrical objects.

Clem Padin

Clem graduated from the University of Arizona with a physics major and a mathematics minor and has continued studying mathematics ever since. He is pictured next to the little library that he built for his neighbourhood.

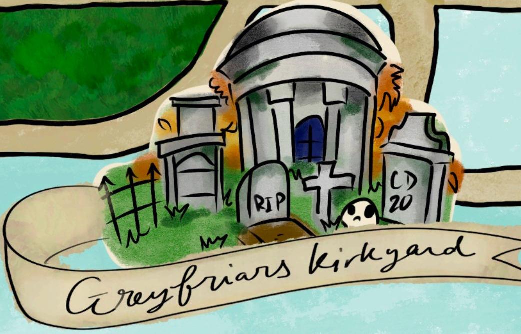
✉ clem.padin@gmail.com

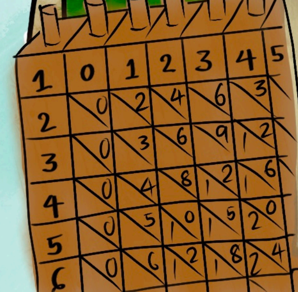
A mathematician's guide to Edinburgh

Remember that Edinburgh is in *Scotland* and so dress for all weathers. Some may choose to wear something with a knot in it, in memory of Peter Guthrie Tait, an early developer of knot theory. Some may knot.

You may be a bit peckish after your travels so be sure to drop in to one of the numerous pubs in Edinburgh and order a pi-e.

Have a rest at Greyfriars Kirkyard: the final resting place of Colin Maclaurin and James Stirling. It may not be the most comfortable bedding... but with such company, who cares!





Dùn Èideann, Auld Reekie, the Athens of the north...

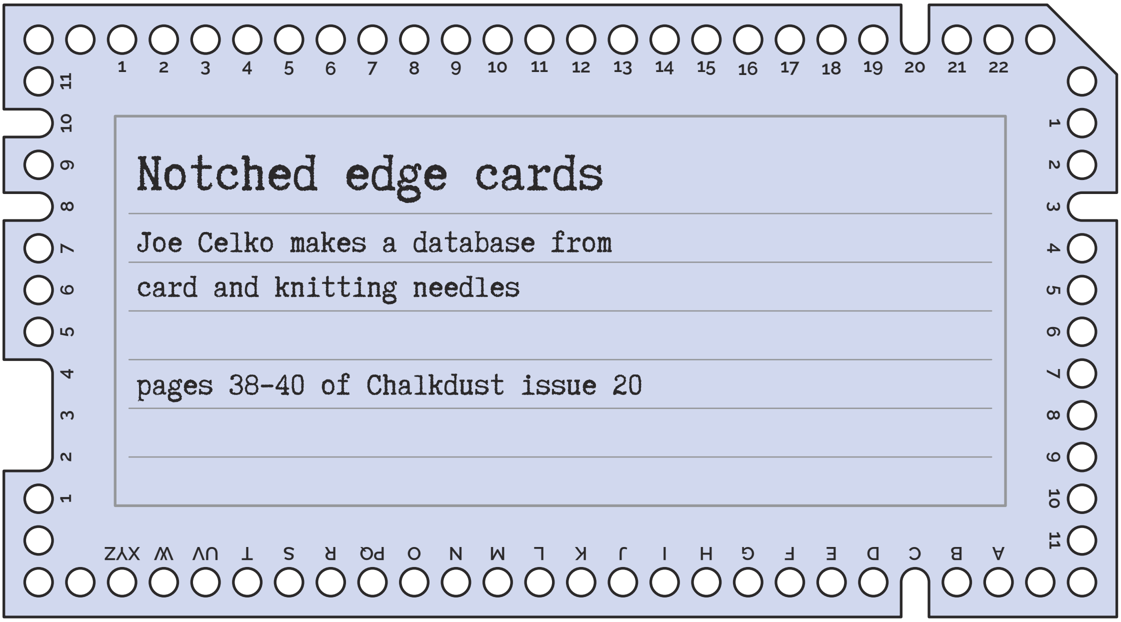
Edinburgh has had a few different names and personalities, but have you ever considered its mathematical side?

Grab a photograph with James Clerk Maxwell, inventor of the theory of electromagnetism. His statue is located next to the railway station. Make sure you tell your non-mathematical friends that without his curls and divs you couldn't take that colour photograph!

Take a quiet stroll up Calton Hill to take in the Old Observatory. To really get the most out of the movements of the planets, why not go that extra mathematical step and read semi-local legend (and first ever scientist) Mary Somerville's book on celestial mechanics while you're there.

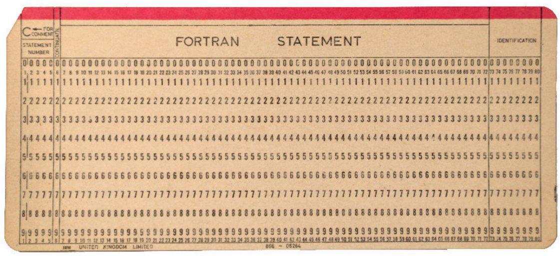
For an evening's entertainment, find the sign reading *He who is without mathematics shall not enter*. This is conveniently around the corner from the pub and dance venue Stramash: visit on a Wednesday to take part in the free evening ceilidh—a traditional Scottish country dance full of algo-rhythms.

If you want to see even more maths in Edinburgh, we recommend Benjamin Goddard's mathematical walking tour. It has multiple variants of different lengths and even includes puzzles!



↑ A notched edge card for this article with notches clipped for the first letter of the author's name (bottom), the number of pages in the article (right), the issue of *Chalkdust* (top), and the categories that the article fits into (left)

↓ A punch card, ready to store one line of Fortran77

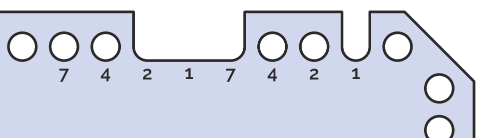


ONCE upon a time there were no cheap personal computers, which made storing data and looking up information really slow. But there was a solution: *notched edge cards*. Much of the terminology and techniques for these cards were borrowed from expensive mechanical technology used by regular punch cards.

But what is a notched edge card? Imagine a stiff piece of card with nice small round holes punched in the edges. A handheld punch—which looked like the punches used at one time for various paper tickets—can be used to turn a hole on the edge of the card into a notch, hence the name of the cards.

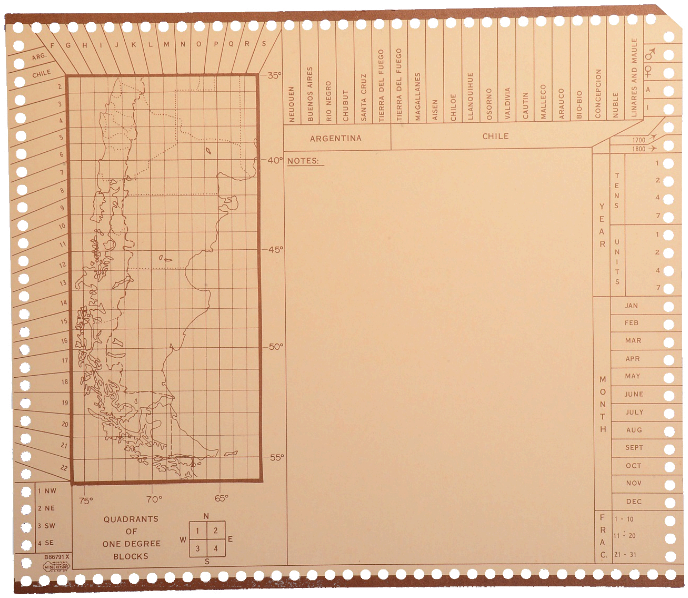
Once all the notch making is done, the cards are stacked in a deck. Borrowing a trick from punch cards, the upper right hand corner of each card is usually clipped off, allowing you to more easily make sure that all of the cards in a deck are facing the same way.

Long metal 'knitting needles' can then be inserted through the holes. With a little shaking, the needle is lifted up, and the cards that did not have a notch lift out on the needle. The notched cards are squared off and put in front of the unnotched cards. This process can be repeated with multiple holes, and at the end the cards with the same pattern of notches on their edges will be grouped together. With



a little practice, anyone can quickly put the deck in sorted order.

This technology would not have been good for a very large database, but for several hundred cards it was wonderful. It required no expensive equipment or even electricity! The handheld notching punches originally cost less than around £5 each, and you can still find them for around £10. The original sorting needles had wooden and later plastic handles; these handles are not really needed, but can easily be replaced with handles from the hardware shop tool section.



↑ A McBee Keysort B86791X card, designed for storing data about birds in South America

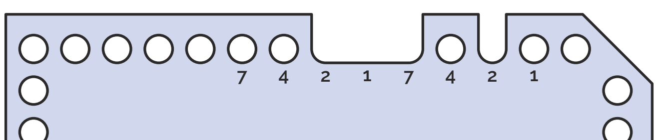
When the notched edge cards were popular, and being used by larger commercial enterprises, there were some other tools available. There was a bulk notch tool, which could clip a notch in the same place in several hundred cards at once—think of it as a specialised office paper cutter. There was a vibrating platform which could help stack and square off a deck of notched edge cards. But the most interesting and rare piece of hardware was the card punch from the Royal McBee company for their Keysort cards. You could not buy one and instead had to rent it from the company, a practice started by IBM for the early unit record—that's what we called punch cards back then—equipment.

One notched edge card is placed in a slot at the top of the punch, and below that are several columns of buttons. The keys in each column were numbered 7, 4, 2, 1. The operator pushed down these keys which locked in place. When the operator was happy with the configuration of punches they had encoded for that edge, they pulled the lever on the side the device, and all the punches came up and notched the columns on that edge of the card in a single stroke. You proceeded in this fashion with the rest of the cards in the deck. When you had finished one edge, you could rotate the cards to notch the holes on the next edge.

Many of the early adding machines also had this particular mechanical design: you push down keys to input the numbers you wanted to add, then pulled the lever and that number was printed on a continuous roll of paper. By throwing some other switches, you could get the total output onto the paper.

The best selling brand was the McBee Keysort, probably because they came up with the support devices that I just discussed and they pre-printed cards for common business applications. For example, it was possible to get blank multipart cheques pre-printed and pre-punched. This meant that a small business could reconcile its cheques without buying expensive tabulating equipment.

Keysort cards became immensely popular for public



schools and small-town libraries in the United States. They also gained popularity for military occupational specialties in the US army, record keeping in the US forestry service and other locations that did not have a lot of electrical outlets or big budgets.

There were also notched edge cards that had space in the centre for microfiche—a photographic technique that takes very small pictures of documents to save storage space and speed up access. By mounting them on a notch edge card, the operator could quickly find any given document they desired. I'd like to remind you the technologies I'm discussing are very old, so don't be too judgmental—but these cards were still in use into the mid- and late 1960s.

Encoding systems

The simplest way to encode digits on notched edge card was group four holes into what was called a field. Within the field, the holes represent the values 7, 4, 2 and 1 from left to right. To store a digit in a field, you simply notched one or two holes. You stored a value of 3 by notching both 2 and 1. Likewise, $5 = 4 + 1$, $6 = 4 + 2$, $8 = 7 + 1$ and finally $9 = 7 + 2$. It is possible to store a 7 by notching 4, 2 and 1 (instead of just notching the 7), but this involves three columns and physically weakens the card, so was avoided. Likewise, you could store some numbers larger than 9 by notching more than two holes, but since we use a base 10 number system it was more convenient to store each digit of a number separately.

Letters can also be done in two punches each. The trick is that some combinations will represent multiple letters, much like you see on some paper index products which will have 'XYZ' on a single tab.

Finally, we could use what was originally called a *superimposed code*. The most popular of these was known as *Zatocoding*, but today is referred to as *hashing* and we have a better understanding of the mathematics behind it. While the encoding of letters and digits was relatively standard, each of these hashes was unique to the set of data we were trying to index, and required a copy of the key to the hash. It's still a good technique though, especially if you have a small data set that's relatively static (notice that the definitions of 'small' and 'static' have changed a good bit since this technology was popular).

Making cards

If you want to make your own card database, it's not too hard to get hold of what you need. Knitting needles that can be used as sorting needles are easy to get from any craft shop. Notching punches can be found on eBay, or you might get lucky and find them in an office supplier, maybe sold as a ticket punch. Getting the cards themselves is the real challenge.

I recommend finding a print shop that does wire spiral binding: the punch they use to make the holes for the spiral bind often makes circular holes around the right size and distances apart. You'll want to print on your cards before sending them to the bindery: it's hard to print pages that have already been punched.

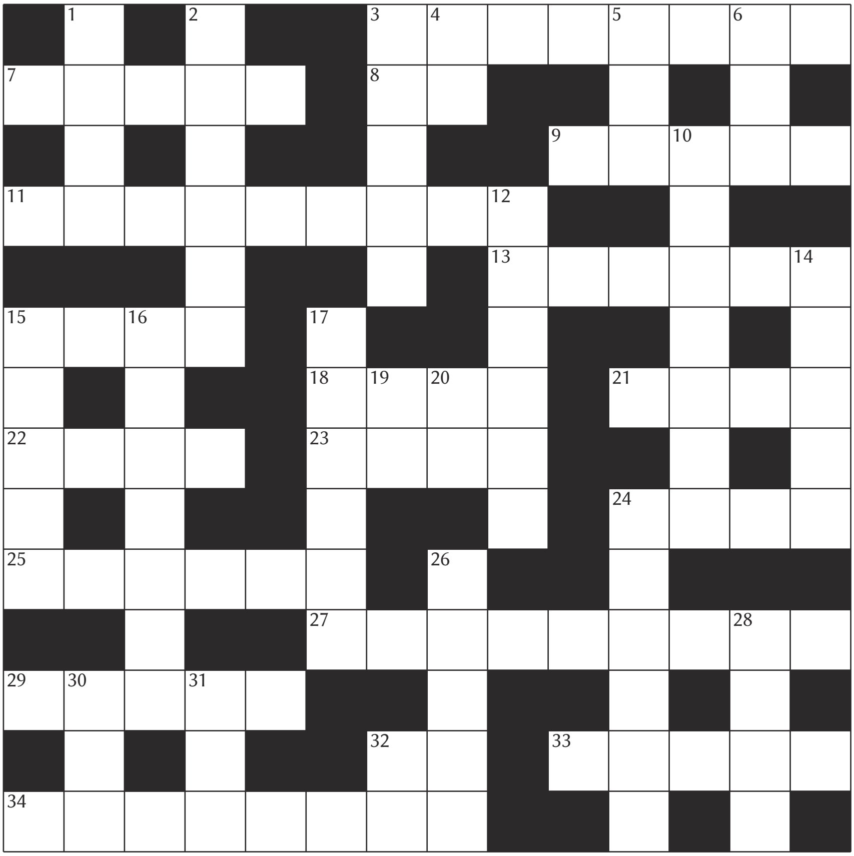
I hope you get a chance to play with this technology and it gives you some incentive to read about the history of early computing equipment.

Joe Celko

Joe is best known for his work with SQL and relational databases, but he was a mathematician first and loves weird recreational things.

Cryptic crossword

#8, set by **Seuss**



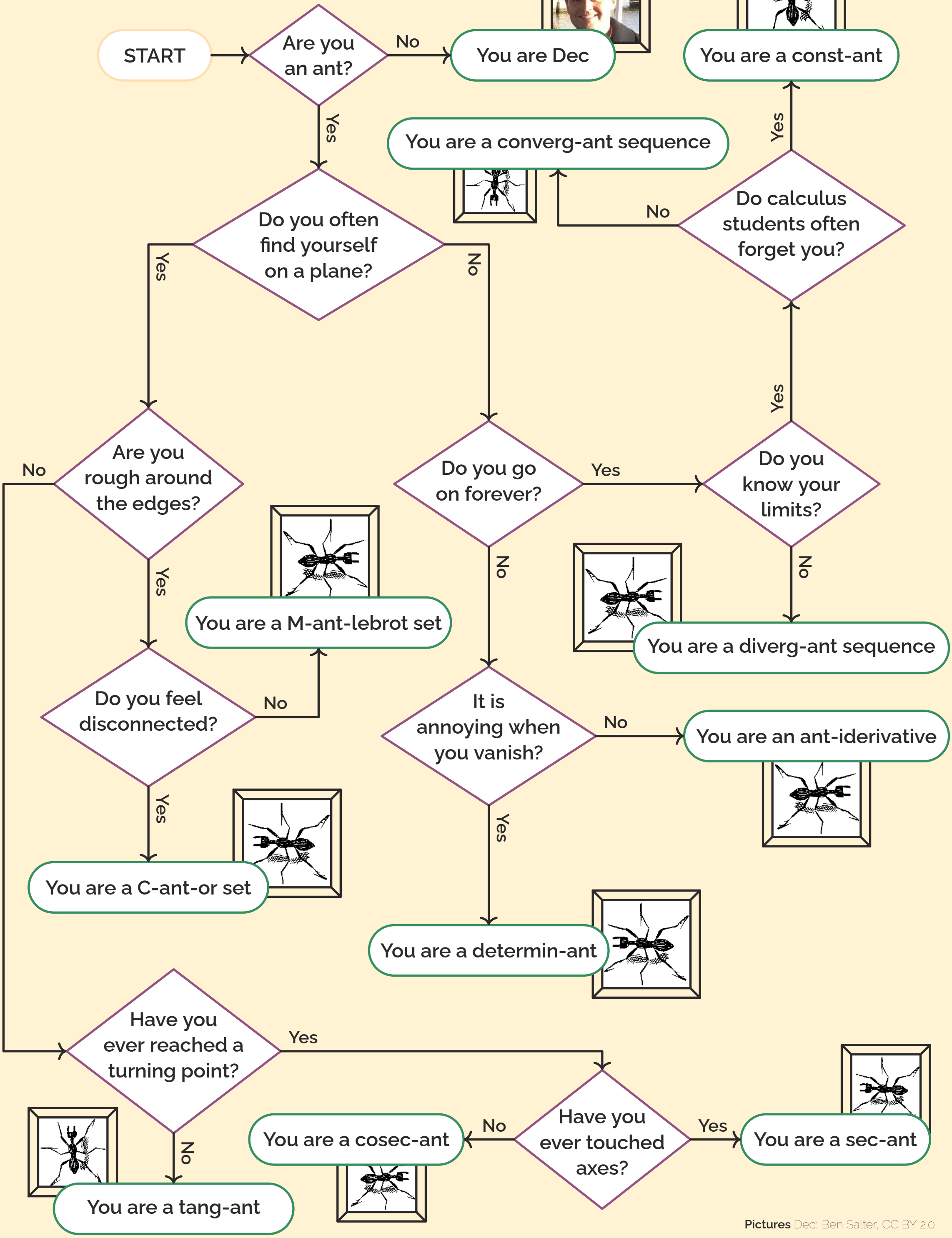
Across

3	Debate the messy garment around uniform.	**(8)**
7	Drunk sort, you donut!	**(5)**
8	Flip heart to get third element.	**(2)**
9	Southern couple goes about introducing quirky rule to code square roots.	**(5)**
11	Slack organisation leads the (definitely humorous) university magazine.	**(9)**
13	Heartless warning of the problem that doesn't 'add up'.	**(6)**
15	Absurd ending to root expression.	**(4)**
18	Not down, not under, onto!	**(4)**
21	Kitten half-ate last shape.	**(4)**
22	`while cryptic != solved:` `go round in circles.`	**(4)**
23	Trolley intercepted by Descartes.	**(4)**
24	Example: initially, grab some items in one basket.	**(4)**
25	Lure into tangled net with frost.	**(6)**
27	Sangwin's shapes, strangely, can reverse derivatives.	**(9)**
29	Itchy reeler cancels odds at half six.	**(5)**
32	$\mu = \rho V(v - at)$.	**(2)**
33	Cat beheaded after hospital spiral.	**(5)**
34	Base secret plan for my data table.	**(4,4)**

Down

1	A/hh! A hyperbola!	**(4)**
2	Del backed into 3/4 of curb. Del is crossed!	**(6)**
3	Record in fixing lumbar contraction.	**(5)**
4	Christmas lectures... during Christmas?	**(2)**
5	Milliequivalent of mass, energy and charge.	**(3)**
6	Hot or not: a reversal gate.	**(3)**
10	Engineer I rang is a ladder operator.	**(7)**
12	No issues, went inside, thank you!	**(6)**
14	"Eek!" (Goes outside around nerds).	**(5)**
15	Butcher loves to work out.	**(5)**
16	Rest, o-or cuckoo farm alarm?	**(7)**
17	Atomic centres alter nil cue.	**(6)**
19	Personal assistant measuring pressure.	**(2)**
20	Logic operation giving work-core.	**(2)**
24	Ned follows bear, gets decapitated, deserved.	**(6)**
26	Malty beer put out next to street.	**(5)**
28	Place to hide in Guatemala, ironic!	**(4)**
30	Hot or not: a Chappell Roan song starting.	**(3)**
31	A tree to reclimb regularly.	**(3)**
32	Odd muon shortens the moment.	**(2)**

Which animal are you?

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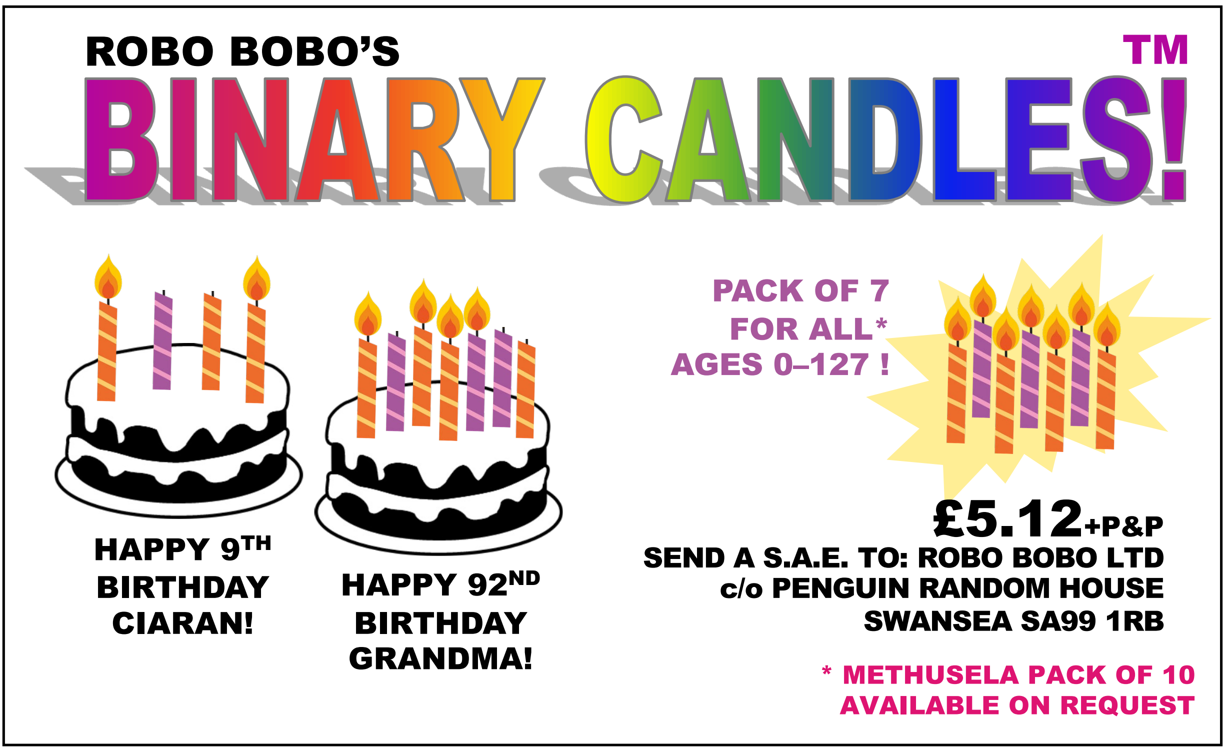
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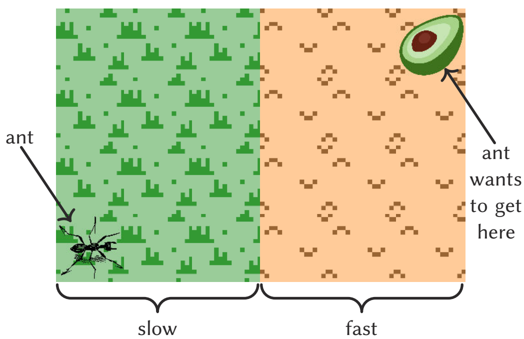
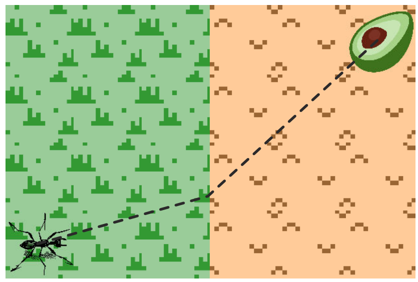
Ferm-ant's principle

HERE'S a standard(ish) calculus question, known as Fermat's principle: what is the quickest way to get from one corner of a field to the opposite one, if one side has been mowed so you can move speedily across it, but the other side is growing wild and slowing you down?

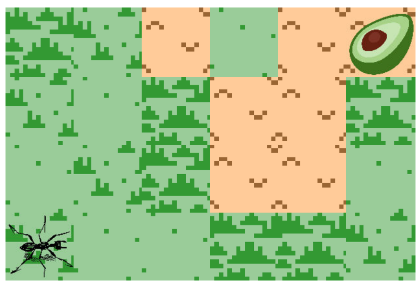
If you set off straight towards the opposite corner, you'll take the shortest path—but not necessarily the quickest one. The fastest path will involve travelling a bit further on the nice ground, and a bit less far on the overgrown side; it might look a bit like this:



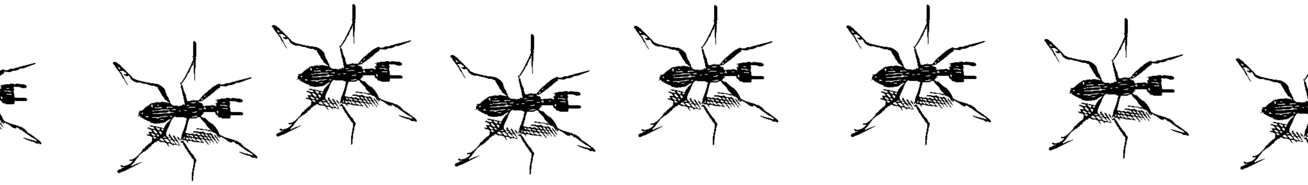
You could sit down and work it out (it's not too tricky), using a bit of stationary point analysis. Or, you can get ants to solve it for you! Just put a colony of hungry ants in one corner of the field and some food on the other. The foragers will want to collect the food and bring it back as quickly as possible. Some researchers from Regensburg in Germany actually tried it out, and the ants found the quickest route between the two corners, taking into account the different speeds they can move on different surfaces. Thanks, ants. Thants.

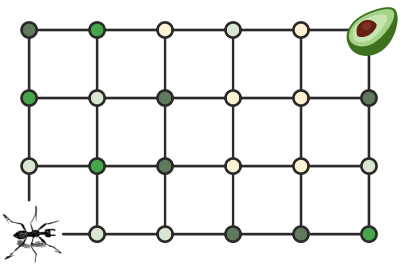


We can make the picture a bit more complicated: instead of the field being mown on one side and overgrown on the other, we could assign a different grass height to each point in space. (Mathematicians call this a… field.)



The ants will still want to take the fastest path from one side to the other, but now the calculus problem is starting to get complicated. It can help if we split the field up into sections, and draw a graph representing all the different routes the ants could take.





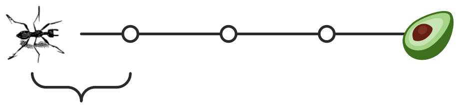
Once we know how long it will take the ants to move along each edge, we *just* have to check all the routes and figure out which one is fastest.

Prob-ant-bility

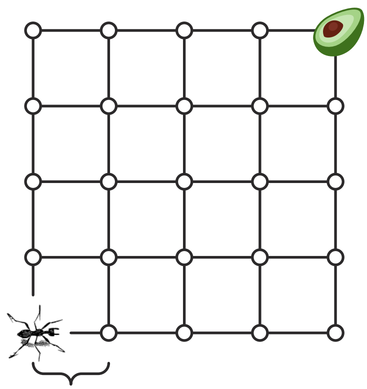
Probability theorists like to ask a very similar question, but about *random* fields. Instead of fixed timings on each edge, they make each one random. Now that the shortest path is also random, it's natural to wonder what the path is *typically* like: how long is it on average? Does it start by going upwards or rightwards? Is it mostly straight or very wiggly?

There are many types of grass, and so many different choices for the random amount of time it takes to cross an edge. You could choose your favourite distribution. For simplicity though, let's say that the time is a uniformly random amount between one and two minutes.

If our graph is one-dimensional (in other words, a line), there's only one possible route from left to right.



When there are n segments, the average time to get across the field is close to n times the average length of time per segment—so $3n/2$. This is the law of large numbers.



But when we have an $n \times n$ grid, we can definitely get from corner to corner in time $4n$, and it must take at least time n. We know that there exists a number $\alpha \in [1, 4]$ such that the total time is very close to αn, but unlike the one-dimensional case, no one knows what α is!

Tyler Helmuth
Tyler is an associate professor of probability at the University of Durham. Mathematically, he is interested in statistical mechanics. Away from the office he's usually running. Tyler is one of the organisers of the UK Easter Probability Meeting, taking place at Durham University from 31 March to 4 April 2025. See tinyurl.com/UKEaster25 for more details.

There are lots of statements in maths that are easy to state, but hard to prove. This issue, Sej Patel explains...

The Collatz conjecture

★ ☆ ☆

Easy to state

THE Collatz conjecture is an unsolved problem in mathematics. It is one of the most famous, and very easy to state, but hasn't been proven despite being conjectured nearly 90 years ago.

Take any starting integer a_0, and follow this process:

$$a_{i+1} = \begin{cases} a_i/2 & \text{if } a_i \text{ is even,} \\ 3a_i + 1 & \text{if } a_i \text{ is odd.} \end{cases}$$

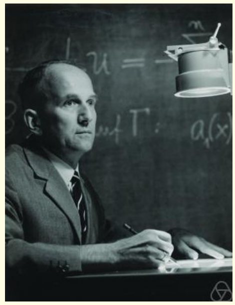
We stop applying this when we get $a_k = 1$ for some k.

For example, with $a_0 = 17$, we get the sequence $17 \to 52 \to 26 \to 13 \to 40 \to 20 \to 10 \to 5 \to 16 \to 8 \to 4 \to 2 \to 1$.

It is conjectured that for any starting number, the process will terminate at 1 after finitely many steps: this is called the Collatz conjecture.

In some places, the statement of the conjecture doesn't include the termination step, and instead says that it will reach 1 after finitely many steps. This is the same thing, since if you don't stop as soon as you reach 1, you'll follow the pattern $1 \to 4 \to 2 \to 1 \to 4 \ldots$ forever.) This conjecture was proposed in 1937 by Lothar Collatz.

The conjecture is widely believed to be true, and so far it has worked for every number mathematicians have tried. Although 'try everything and see what works' is a valid approach (*Chalkdust* has been doing it for years!), it won't give us a rigorous proof that the conjecture is true, as there would be an infinite number of integers to check.



↑ Lothar Collatz

If the conjecture were false, it would mean there is some number whose sequence doesn't terminate at 1. Either it would keep increasing beyond any threshold we set, or it would end up in some loop (like the $4 \to 2 \to 1$ loop) of large numbers.

We could try to get a computer to find a counterexample, which would either end up in some other loop of large numbers or find a sequence which gradually increases forever without looping. However, this lands us back in the issue of asking the computer to carry out an infinite number of calculations in a finite amount of time.

Another approach would be to try to get a general proof which may only need some cases checking—for example, we know all powers of 2 eventually terminate. Can we categorise other numbers into finitely many categories which always terminate? Terence Tao had a go at this in 2019, and proved that the Collatz conjecture is 'almost' true for 'almost all' starting numbers. (In a similar vein, 'almost all' issues of *Chalkdust* have some 'almost' good jokes.)

★ ★ ★ Hard to prove

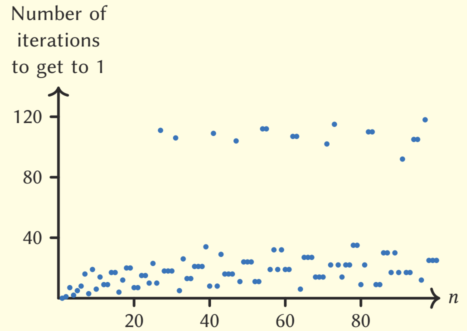
The sequences produced by the algorithm are known as hailstone sequences, due to their rise and fall—until you reach some power of 2, you'll always have to increase at some step when you've reached an odd number.

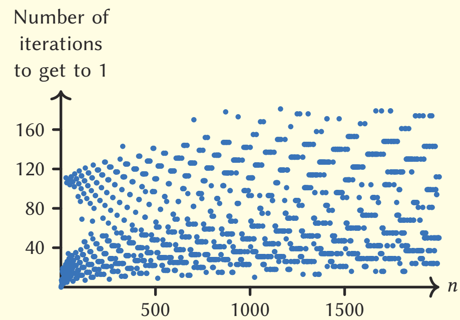
We say that the stopping time of a number is the smallest number of steps needed to reach 1. The stopping time varies for each starting number; if we begin with 32, we reach 1 after 5 steps, but starting with the smaller number of 27 takes 111 steps to reach 1. The graphs below show the stopping times for the numbers up to 100 and 2000.

An easier 'problem', similar to the Collatz conjecture, involves using $a_{i+1} = a_i + 1$ when a_i is odd. This time we get a $1 \to 2 \to 1$ loop, and for every other odd number a_{i+2} will always be smaller than a_i. We can rule out both the 'increasing forever' *and* 'infinite loop of large numbers' cases, so we know that sequences will always terminate.

This doesn't work for the actual Collatz conjecture, because $(3a_i + 1)/2$ is larger than a_i, and if this is odd, then we'll grow larger still, so there's a chance we can keep growing by a factor of $3/2$. However, this chance is small, so probabilistically, we'll decrease in the long run. Unfortunately, this isn't quite enough to prove we'll always reach 1—unlikely things happen all the time!

As of 2020, all initial values up to $2^{68} \approx 2.95 \times 10^{20}$ have been checked, and end at 1. This number has improved from earlier verifications; in 1992 the bound was 5.6×10^{13}, which was improved to $19 \times 2^{58} \approx 5.48 \times 10^{18}$ in 2008.





Sej Patel

Sej has a keen interest in combinatorics, and if she's not thinking about graphs then she's probably reading.

The problem with addition

It doesn't add up for Patrick Creagh

ADDITION is easy, it's one of the first things we teach to primary school kids. $1+1=2$, $2+2=4$, etc. A much more difficult operation to learn is multiplication. The numbers get bigger much more quickly, there are whole tables to memorise and there's something I still don't quite believe about $7 \times 8 = 56$. However, precisely because of multiplication being 'more restrictive' an operation than addition in some sense, many of the biggest unsolved problems in maths are difficult, at least in part, because of addition.

The Goldbach conjecture asserts that any even number bigger than 2 is the sum of two prime numbers and the conjecture is still unsolved. If the problem were instead asking about representing integers n as products of primes, we would quickly see that multiplication is too restrictive, and we can always find some large n that can only be written as a product of any number of primes we desire. Even partially solved additive problems are unwieldy. The partition function of a positive integer (denoted $p(n)$) is the number of ways n can be written as a sum of positive integers (up to reordering); so

$$\begin{aligned} p(1) &= 1: \quad 1 = 1,\\ p(2) &= 2: \quad 2 = 2,\ 1+1,\\ p(3) &= 3: \quad 3 = 3,\ 1+2,\ 1+1+1, \end{aligned}$$

and so on. However, no closed form expression for $p(n)$ is known and the best we can do is a recurrence relation or approximate $p(n)$ (which 'looks like'

$$\frac{1}{4n\sqrt{3}} \exp\left(\pi\sqrt{\frac{2n}{3}}\right)$$

for large n). However, if the problem was instead: how many ways a positive integer n can be written as a product of integers (denoted by say $\tilde{p}(n)$), then $\tilde{p}(n)$ will be smaller in general than $p(n)$ and depends only on the prime decomposition of n (which admittedly has a whole host of other problems). "But maybe these two problems are just outliers!" I hear you cry. Let's slowly build up to an additive problem called Waring's problem and showcase some of the techniques used to tackle it by first looking at how a similar multiplicative problem might be dealt with.

Given some positive integer n, can we find non-negative integers x and y which satisfy

$$x^2 \times y^2 = n?$$

To solve this problem we can use a trick: $n = (xy)^2$, so n must be a square number. Say $n = m^2$ for some positive integer m so we can re-write

$$x \times y = \sqrt{n} = m.$$

Now we can appeal to the fundamental theorem of arithmetic, which tells us that m can be written as a unique product of prime numbers, up to reordering. Thus, any x and y must themselves be products of these primes also. Then, after a simple counting exercise, we can even find all possible pairs x and y which solve this equation. Even if we throw more variables than just x and y into the mix, we will only ever find solutions where n is a square number.

This technique will also work for powers other than 2, or for problems with more variables than just x and y. In general, for a power k and s-many variables $x_1, \dots, x_s$, the equation

$$x_1^k x_2^k \cdots x_s^k = n,$$

has solutions only when $n = m^k$ for some integer m. In this case, exactly as before, the fundamental theorem of arithmetic tells us there exists some (not necessarily distinct) primes $p_1, \dots, p_t$ such that

$$x_1 x_2 \cdots x_s = m = p_1 p_2 \cdots p_t.$$

To find all solutions we can set each x_i ($i = 1, \dots, s$) to some product of the p_j ($j = 1, \dots, t$), ensuring that each p_j occurs for only one of the x_i. This is easy to do precisely because of how 'restrictive' multiplication is compared to addition and it shows just how powerful a result the fundamental theorem of arithmetic really is.

Now let's change this from a multiplicative problem to an additive one, and ask the same question. Given some positive integer n, can we find non-negative integers x and y which satisfy

$$x^2 + y^2 = n? \qquad (❧)$$

Suddenly, and despite using the 'easier' operation of addition, this has become a much more difficult problem to solve. It's not even obvious if there are solutions for every n. As always, a good place to start when stuck is to try some test cases. Cases $n = 1$ and $n = 2$ are both straightforward:

$$1^2 + 0^2 = 1, \qquad 1^2 + 1^2 = 2.$$

But what about $n = 3$? We can't have x or $y \geqslant 2$ as then the left hand side of equation (❧) would be at least 4, a contradiction. We are stuck again.

Maybe if we allow more variables we can find solutions for more numbers n? Maybe, 3 variables is enough. So the question becomes: given some positive integer n, can we find non-negative integers x, y, z which satisfy

$$x^2 + y^2 + z^2 = n?$$

Now, we can find solutions for all n up to 7. But, similarly to $n = 3$ in equation (❧), 7 itself poses another problem. Maybe given a positive number n we need four square numbers to be able to write n as a sum of squares, ie

$$x^2 + y^2 + z^2 + w^2 = n.$$

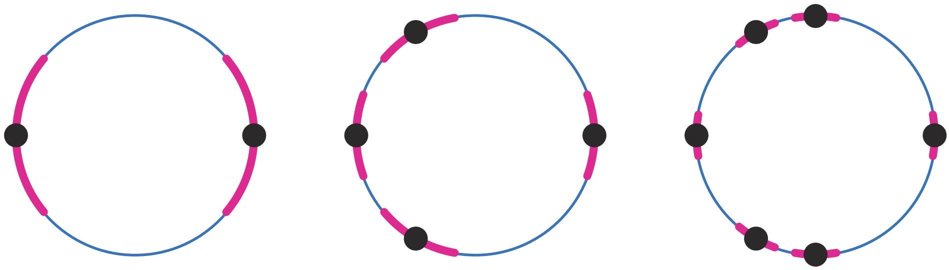
This now seems promising as we check more and more integers n we can keep finding integers x, y, z, w which satisfy this equation. In fact, Lagrange proved in 1770, by looking at remainders of numbers and their squares when divided by primes, his four-square theorem, which guarantees that 4 variables is enough for any positive integer n we choose.

CHALLENGE

Find a way to represent 999 as a sum of four squares.

Hint: you will need to use four nonzero squares and there are actually 27 different ways to do this!

Phew! That was a lot of work compared to the multiplicative case where the fundamental theorem of arithmetic solves all our woes. To make matters worse the techniques of looking at remainders of cubed numbers and higher powers aren't as nicely behaved as squares, so, the method to prove Lagrange's four-square theorem breaks down for these higher powers. We've only answered the question for squares! If we allow any power and as many variables as needed, we can state the aforementioned Waring's problem in full generality.



↑ The intervals for Waring's problem with $n = 2^k$, 3^k, 4^k respectively on unit circles in the complex plane are shown by the sections in pink, which are the points $\mathrm{e}^{2\pi i\alpha}$ satisfying $|\alpha - a/q| < n^{-1+1/(5k)}$ for each $1 \leqslant q \leqslant n^{1/k}$ and $0 \leqslant a/q \leqslant 1$. The intervals with smaller contributions are the leftover sections in blue. Notice how the intervals shrink as there are more and more disjoint intervals. In reality the pink sections are much much smaller than what is shown here but then they would be too difficult to see!

Waring's problem

For a given integer k, is there some integer s such that given any positive integer n we can find non-negative integers $x_1, x_2, \ldots, x_s$ satisfying

$$x_1^k + x_2^k + \cdots + x_s^k = n?$$

To try and solve this problem we can appeal to techniques from a branch of maths called analytic number theory.

The goal for the problem (and indeed much of analytic number theory) is to find an asymptotic formula, which is an approximation for some desired function that gets better and better with bigger and bigger numbers, then manually check the finitely many numbers leftover which aren't approximated well. For Waring's problem, we define the function $r_{k,s}(n)$ to be the number of solutions $(x_1, \ldots, x_s)$ to the above equation. Waring's problem can now be equivalently stated as: Given some k, is there an s such that $r_{k,s}(n) \geqslant 1$ for all positive integers n? But, we still need a clever trick to rewrite $r_{k,s}(n)$ in a way that is useful to us. This clever trick comes from a large branch of mathematics called complex analysis.

Complex analysis is a branch of mathematics largely concerned with the theory of functions of complex numbers, but all we need are some results about integrating complex functions in the complex plane. Given some n we can take the seemingly arbitrary function

$$f(z) = \sum_{m=0}^{N} z^{m^k},$$

where we require $N \geqslant n^{1/k}$ and we finally get the integral:

$$\begin{aligned} r_{k,s}(n) &= \frac{1}{2\pi \mathrm{i}} \int_{|z|=1} \frac{f(z)^s}{z^{n+1}}\,\mathrm{d}z \\ &= \int_0^1 f(\mathrm{e}^{2\pi \mathrm{i}\alpha})^s \mathrm{e}^{-2\pi \mathrm{i}\alpha n}\,\mathrm{d}\alpha. \end{aligned}$$

Some of you may recognise this as the formula for computing the Fourier coefficients of the periodic function $f(\mathrm{e}^{2\pi \mathrm{i}\alpha})^s$.

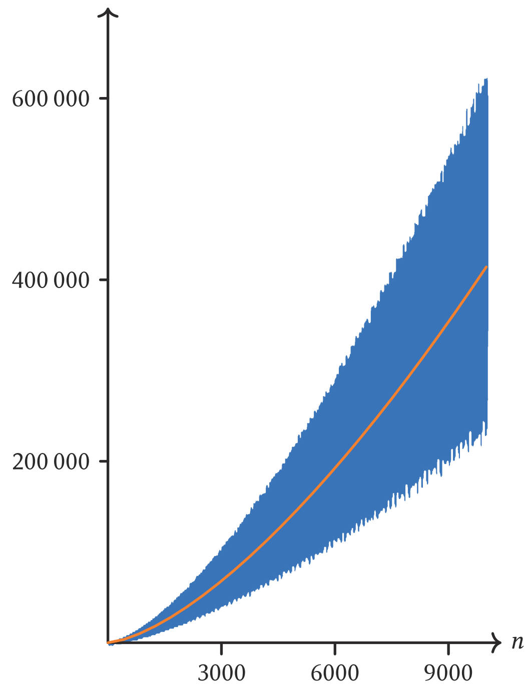
The rest of the work is finding suitable bounds and estimates for this integral. It turns out that most of the size of the integral comes from small intervals around rational numbers between 0 and 1 with small denominator. Everything else has a relatively small contribution to the integral. An example of what these intervals look like on the unit circle can be seen above.

Eventually, after approximating the integral on these small intervals, we ultimately find that for 'sufficiently large' n, if $s \geqslant 2^k + 1$ then $r_{k,s}(n)$ behaves like $Cn^{s/k-1}$ for some positive constant C, as seen on the next page.

Or, to put it another way, for sufficiently large n, $r_{k,s}(n) \geqslant 1$ and there is at least one solution for every n.

Circle method

Under the hood of the proof of Waring's problem is a technique known as the circle method. Attributed to Hardy, Ramanujan and Littlewood, this method is used by analytic number theorists to solve a wide range of problems. The crux of this method is to find the Fourier coefficients of the function or series in question. This translates the problem from one about, in this case, the integers, to a problem about a sequence or function around a circle. The hope is that there are separate regions around the circle where the evaluation of the function is small (known as the minor arcs) and large (known as the major arcs). The minor arcs, which will form most of the circle, can then be bounded. This leaves the major arcs, where the contribution is most significant, to be studied. This will often result in a more tractable problem.



↑ The values of $r(n)$ (blue) and $Cn^{1.5}$ (orange) for $k=2$, $s=5$. For large n, $r(n)$ is bounded between approximately $Cn^{1.5}/2$ and $3Cn^{1.5}/2$.

This is certainly a far cry from the simple method we came up with for our multiplicative problem! But we have our general method that works for any integer k. This is an example of a powerful tool called the circle method, which has been used to solve many problems in number theory.

Remember Goldbach's conjecture mentioned at the start? There's a similar, weaker problem which has been solved by the circle method called the ternary Goldbach problem, which claims every odd number bigger than 5 can be written as a sum of three prime numbers.

It should be noted however that there are many shortcomings to our work here. The keen-eyed reader may have noticed that at no point did we find a minimal s for each k (and often we massively overshot the minimum). It is also extremely difficult to specify what 'sufficiently large' actually means and even when we can the estimate is often in the region of $10^{10,000,000}$. Another drawback is that we are limited by how well we can estimate various integrals along the way, which is often not an easy thing to do. Thus the circle method stands as a warning to all despite popular perception, addition is hard!

Patrick Creagh

Patrick is in the first year of his PhD investigating aspects of the ℓ-adic Langlands programme (which tries to show two specific numbers being the same isn't a coincidence) at Durham University. To escape from maths he spends his free time counting rests while playing trombone.

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REVIEWS

Books, films, games, anything mathematical...

Talking Maths in Public podcast

As big fans of talking about maths and of TMiP as an organisation, we were delighted when this podcast started up. Some of our editors did appear in the first episode discussing the 2023 book of the year, but we are definitely not biased, honest... All of the episodes that have been released so far have been full of great segments, from interviews with maths communicators to the obtuse angles where maths communicators get meta when discussing maths communication.

We think that all the episodes are worth a listen.

Gifted

Proud to see Navier–Stokes finally getting the stardom it deserves.

Mean Girls (2024)

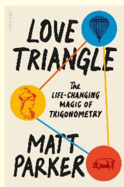
Though I am a fan of recursion, a film of a musical of a film was an iteration too far this time.

Mean Girls (2004)

The perfect mix of teen movie and maths.

Electromagnetic Field 2024

The best festival imaginable.



Love Triangle

Matt Parker

Good on the triangles but not enough romance for a book with love in the title.

Recreational mathematics and its history: In memory of David Singmaster

A one day event all about the eminent recreational mathematician. Katie Steckles's talk about Rubik's cubes was excellent.

The Cliffs & Richards calendar 2025

cliffsandrichards.com

Even worse than last year's.

Borwein integrals

Aimen Khan ponders how patterns break down

THE study of mathematics has long been steadfastly driven towards patterns. They seem to appear out of nowhere, and defy all sense and reason; we depend on the minds of curious thinkers to give them any logical backing at all. After all, human brains are designed to look for order and structure in our environments to predict what will come next. It gives us an edge over those that can't. Stocking grain for winter becomes incredibly useful when you know that you cannot harvest wheat in December, and following herds is a lot easier when you know that they move up north for summer every year.

Mathematicians function in much the same way—we can't deny that we do have biases towards pretty and unexpected results! Little mind is given to the patterns that don't fit into our box of mathematical tricks, and we overlook what beauty they can hide in their spontaneity. Sometimes the patterns that break hold new secrets entirely, and show us new ways of viewing problems that may not have been apparent in the first place. But the last place we expect to see something like this is a normal looking integral.

That sinc-ing feeling

We can start with the sinc function across the real line, which is commonly used in signal analysis for signal reconstruction and frequency filtering.

$$\operatorname{sinc} x = \frac{\sin(x)}{x}.$$

We can use a nice trick to integrate sinc over the whole real line:

$$\int_{-\infty}^{\infty} \operatorname{sinc} x \,\mathrm{d}x = \int_{-\infty}^{\infty} \frac{\sin(x)}{x} \,\mathrm{d}x.$$

Since both sine and $1/x$ are odd functions, multiplying them gives an even function that is symmetric across the y-axis. This means we can split our integral into two parts, which are both equal:

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{\sin(x)}{x} \,\mathrm{d}x &= \int_{-\infty}^{0} \frac{\sin(x)}{x} \,\mathrm{d}x + \int_{0}^{\infty} \frac{\sin(x)}{x} \,\mathrm{d}x \\ &= 2\int_{0}^{\infty} \frac{\sin(x)}{x} \,\mathrm{d}x.\end{aligned}$$

This is a pretty well known integral, which we can evaluate to be $\pi/2$ using contour integration; so our whole integral comes out as π.

The next term in our pattern comes from multiplying $\operatorname{sinc}(x)$ by $\operatorname{sinc}(x/3)$:

$$\int_{-\infty}^{\infty} \frac{\sin(x)}{x}\frac{\sin(x/3)}{x/3} \,\mathrm{d}x.$$

We can use the same trick of multiplying odd functions to end up with an even function, and once again we end up with π!

Now consider the whole integral class of this form where we have products of $\sin(x/a)/(x/a)$. Our pattern-prone brains immediately latch on to something that seems like it could be the start of a beautiful result in the making. We can even write out our integrals in a lovely triangular pattern:

$$\int_{-\infty}^{\infty} \frac{\sin(x)}{x} \,\mathrm{d}x = \pi,$$

$$\int_{-\infty}^{\infty} \frac{\sin(x)}{x}\frac{\sin(x/3)}{x/3} \,\mathrm{d}x = \pi,$$

$$\int_{-\infty}^{\infty} \frac{\sin(x)}{x}\frac{\sin(x/3)}{x/3}\frac{\sin(x/5)}{x/5} \,\mathrm{d}x = \pi,$$

$$\int_{-\infty}^{\infty} \frac{\sin(x)}{x}\frac{\sin(x/3)}{x/3}\frac{\sin(x/5)}{x/5}\frac{\sin(x/7)}{x/7} \,\mathrm{d}x = \pi,$$

$$\int_{-\infty}^{\infty} \frac{\sin(x)}{x}\frac{\sin(x/3)}{x/3}\frac{\sin(x/5)}{x/5}\frac{\sin(x/7)}{x/7}\frac{\sin(x/9)}{x/9} \,\mathrm{d}x = \pi,$$

$$\int_{-\infty}^{\infty} \frac{\sin(x)}{x}\frac{\sin(x/3)}{x/3}\frac{\sin(x/5)}{x/5}\frac{\sin(x/7)}{x/7}\frac{\sin(x/9)}{x/9}\frac{\sin(x/11)}{x/11} \,\mathrm{d}x = \pi,$$

$$\int_{-\infty}^{\infty} \frac{\sin(x)}{x}\frac{\sin(x/3)}{x/3}\frac{\sin(x/5)}{x/5}\frac{\sin(x/7)}{x/7}\frac{\sin(x/9)}{x/9}\frac{\sin(x/11)}{x/11}\frac{\sin(x/13)}{x/13} \,\mathrm{d}x = \pi.$$

They all equal π! And now the next term…

$$\begin{aligned}&\int_{-\infty}^{\infty} \frac{\sin(x)}{x}\frac{\sin(x/3)}{x/3}\frac{\sin(x/5)}{x/5}\frac{\sin(x/7)}{x/7}\frac{\sin(x/9)}{x/9}\frac{\sin(x/11)}{x/11}\frac{\sin(x/13)}{x/13}\frac{\sin(x/15)}{x/15} \,\mathrm{d}x \\ &= \frac{467807924713440738696537864469}{467807924720320453655260875000}\pi \approx \pi - 4.62\times 10^{-11}.\end{aligned}$$

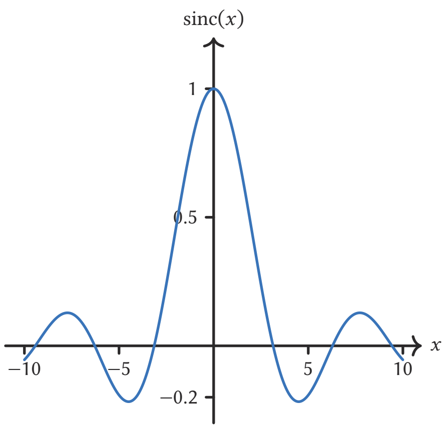
Our perfect pattern reaches an abrupt and rather anticlimactic end. It's just off, but why?

The integrals were originally observed by late father–son pair of mathematicians Jonathan and David Borwein, who first evaluated it with an early model of some integral calculator software in 2001. Naturally, the small discrepancy in the calculation value looked like a numerical error on the side of the integral calculator, which, at the time, was in early stages of development and highly prone to errors. It was brushed off as a mistake and sent to the integral calculator company as a bug, although some stories suggest that Borwein knew the pattern, and sent it in as a practical joke (perhaps mathematicians have a sense of humour after all).

The whole pattern goes off in a completely different direction, getting further and further from π each time.

Everything but the kitchen sinc

To understand how this works, we employ the Fourier transform. Essentially, this deconstructs our function into a sequence of sine and cosine waves, which is really helpful when we want to analyse a function further, or make certain parts of it easier to manipulate.



↑ The sinc function.

To find the Fourier transformation of a function, f, we calculate the integral

$$\mathcal{F}(\xi) = \int_{-\infty}^{\infty} f(t)\mathrm{e}^{\mathrm{i}2\pi\xi}\,\mathrm{d}t.$$

To get the integral of our original function back, we set $\xi = 0$, which makes our exponent term 1.

$$\mathcal{F}(0) = \int_{-\infty}^{\infty} f(t)\cdot 1\,\mathrm{d}t$$

The convolution theorem tells us that if two functions are multiplied in the integral inside a Fourier transform, we can deal with them as a convolution under the right conditions:

$$\mathcal{F}(f_1 \cdot f_2) = \mathcal{F}(f_1) * \mathcal{F}(f_2).$$

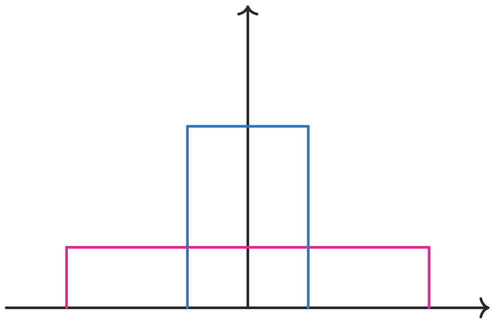
A really interesting geometrical implication of convolving two functions is that we get what we can essentially describe as a 'moving average' of the two functions, which is the result of a weighted blend of the two functions. A popular way to visualise it for the sinc function is as a window whose width is determined by f_1 passing over the graph of f_2, and the average of all the values inside the window at position x becomes the height at x on the graph of the convolution.

So, to grasp what's actually happening to our sinc functions as we multiply them together inside the transform, we need to understand what happens once we take their transforms and convolve them with other sinc functions.

When we decompose our sinc function using a Fourier transform, we get the rectangle function: it's a constant between −1 and 1, and zero everywhere else.

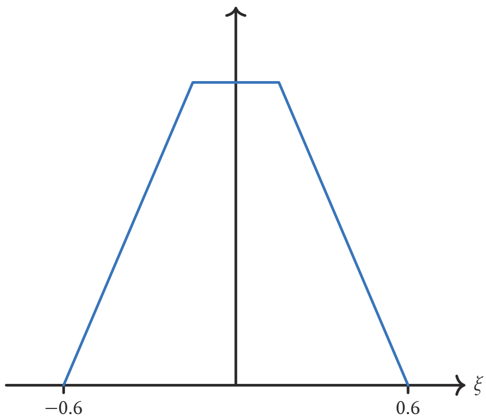
When we transform $\operatorname{sinc}(x/3)$, we get another rectangle that is three times as narrow and three times as tall, preserving the area under it.

$$\mathcal{F}\left(\frac{\sin(ax/k)}{ax/k}\right) = k(\operatorname{rect}(akx)).$$



↑ The Fourier transform of $\operatorname{sinc}(x)$.

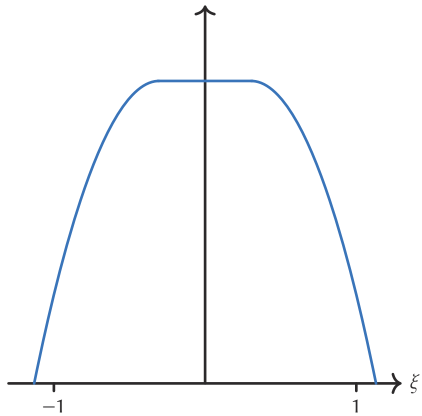
Let's apply the convolution to both our sinc functions, and see where we can go from there. When we blend our two functions together, the width of our window corresponds to $\operatorname{sinc}(x/3)$ and it slides across the rectangle function, averaging out the rectangle and smoothing it into a trapezium.



↑ The Fourier transform of $\operatorname{sinc}(x/3)\operatorname{sinc}(x/5)$

For the next integral, we convolve this trapezium with $\operatorname{sinc}(x/5)$, which translates to taking its moving average with another window. This time its width corresponds to $\operatorname{sinc}(x/5)$, and is a little bit narrower, which takes the graph from the trapezium above to a rounded trapezium with a wider base that still fits inside the $(-1, 1)$ interval from $\mathcal{F}(\operatorname{sinc}(x/3))$, because

$$\frac{1}{3}+\frac{1}{5}=\frac{8}{15}<1.$$



↑ The Fourier transform of $\operatorname{sinc}(x/3)\cdots\operatorname{sinc}(x/15)$.

This works when we bring in $\operatorname{sinc}(x/7)$, because

$$\frac{1}{3}+\frac{1}{5}+\frac{1}{7}=\frac{71}{105}<1;$$

but once we get up to $\operatorname{sinc}(x/15)$, we have

$$\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11}+\frac{1}{13}+\frac{1}{15}=\frac{46027}{45045}>1.$$

Now our graph is suddenly smoothed out so much that it overflows beyond $(-1, 1)$, and spills out of the limits that support the integral being equal to π.

Just one sinc-ular sensation

Now we can see that the odd coefficients that we introduced at the beginning were somewhat convincing red herrings, and this integral isn't alone in its unconventional character, with sudden pattern breaks rather commonplace in other integrals involving similarly structured products of the sinc function, where all the coefficients add to less than one.

In fact, there are many alterations of the integral which allow our pattern to continue for much, much longer before tapering off.

For example, we can simply prepend a factor of $2\cos(x)$ to our integrand to create a pattern that continues up to 113:

$$\int_{-\infty}^{\infty} 2\cos(x)\frac{\sin(x)}{x}\,dx = \pi,$$

$$\int_{-\infty}^{\infty} 2\cos(x)\frac{\sin(x)}{x}\frac{\sin(x/3)}{x/3}\,dx = \pi,$$

$$\vdots$$

$$\int_{-\infty}^{\infty} 2\cos(x)\frac{\sin(x)}{x}\frac{\sin(x/3)}{x/3}\cdots\frac{\sin(x/110)}{x/110}\frac{\sin(x/111)}{x/111}\,dx = \pi,$$

$$\int_{-\infty}^{\infty} 2\cos(x)\frac{\sin(x)}{x}\frac{\sin(x/3)}{x/3}\cdots\frac{\sin(x/111)}{x/111}\frac{\sin(x/113)}{x/113}\,dx = \pi - 4.6648\times 10^{-138}.$$

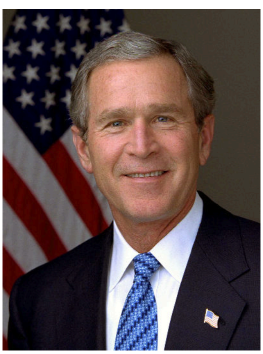
This works because the cosine term increases the overtip value to 2. Our sequence

$$1,\ 1+\frac{1}{3},\ 1+\frac{1}{3}+\frac{1}{5},\ \ldots$$

does (eventually) diverge to infinity, but it does so incredibly slowly, allowing the integrals to equal π for longer.

The hills are alive with the sound of sinc-ing

The physicists Satya N Majumdar and Emmanuel Trizac proposed an alternative way to understand these integrals: by employing 'random walkers'. These lost souls make random jumps in any direction and over any distance, as long as they follow some very specific rules.

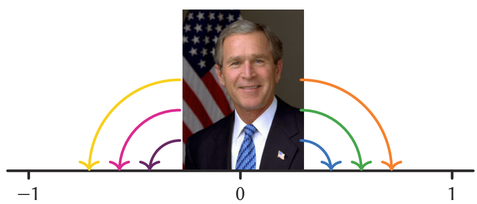


↑ A random George Walker Bush

These models are incredibly helpful when it comes to modelling the randomness of Brownian motion and other particularly volatile behaviour patterns, such as share prices, gene reconstructions, social behaviour, and even Google's PageRank algorithm for sorting websites.

We start our walkers with some basic rules:

1. They all start at the origin of a one-dimensional space.
2. They can move in either direction (left or right).
3. At each step, the maximum distance they can travel is one of our odd coefficients: on the first step, they move a random distance between 0 and 1; for the second, they move a random distance between 0 and 1/3, and so on. Within the interval, the distance chosen is uniformly distributed.



↑ Our Walker taking their first step.

So, on the first step, all the walkers start at 0, and can move anywhere in the interval $[-1, 1]$, and on the second step they can move by a distance up to $1/3$, or negative, towards $-1/3$, and so on.

Now we ask, after the nth step, how many walkers remain close to the origin?

Rather strangely, it seems to behave like the value of our nth Borwein integral.

For the first seven steps, the probability density (the probability of finding a walker at 0) is $1/2$.

This is because some of the walkers hopped right out to the edges, and are close to 1 or -1 with their first step, and it is not possible for them to get back to 0 in only seven steps.

As soon as our walkers have taken more than seven steps, this is no longer the case: some will be able to return from 1 and -1. Other walkers may not return, but instead go further, allowing the group to become more spread out, and the number of walkers near the origin just slightly drops.

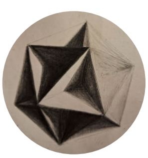
As the number of walkers we start with reaches infinity, the drop in density approaches the value of the *deviation of the Borwein integral from* π at the eighth step.

Just sinc about it

Borwein integrals reveal how patterns can emerge and break in surprising ways, showing the depth and complexity hidden within simple mathematical expressions. We can conjure them from the inner workings of wave transmissions, optical signals, and the whimsical dance of random walkers.

The integrals really are a beautiful reminder of the depth and unpredictability of mathematics. It isn't often we find beauty in a chaotic system, but when we do, it's always eye-opening (as well as just being nice to look at!). Picking apart the maths behind the breaks in patterns like this lift the curtains to let us see behind the scenes of our clockwork universe and pick out the cogs. Perhaps all we can do is hope that we will one day find the manual.

↓ Infinitely many walkers



Aimen Khan

Aimen Khan is a secondary school student from the UK who enjoys exploring patterns, art, and all things geometry. When not stuck into a good puzzle, you can find her out sailing or mountain biking.

Fortune teller

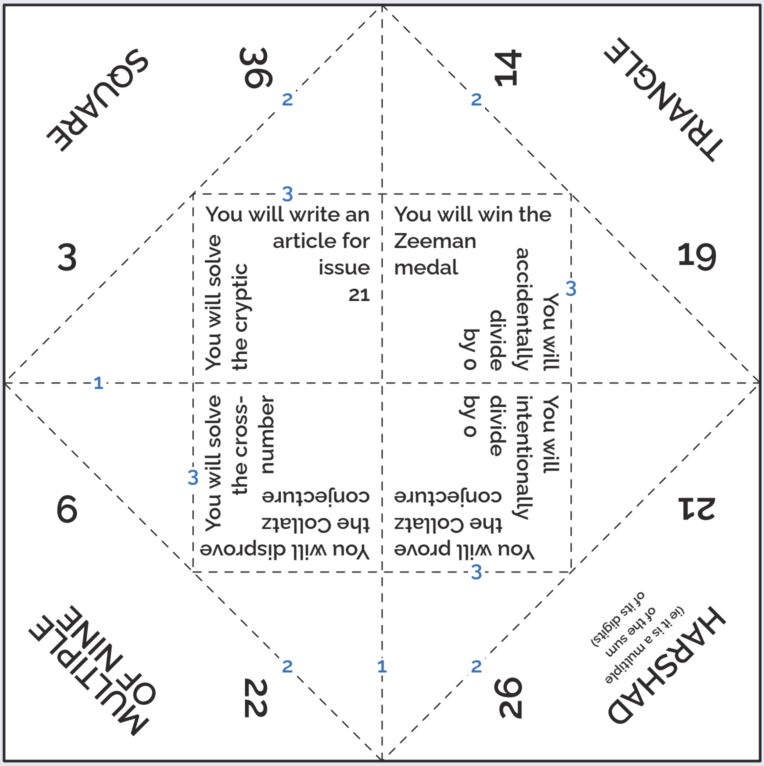
You will need

- Scissors
- This page

These instructions involve two types of fold: a mountain fold and a valley fold.

In a mountain fold, the line you are folding along should be on the outside of the fold (like it's the top of a mountain).

In a valley fold, the line you are folding along should be on the inside of the fold (like it's the bottom of a valley).



Constructing

- Cut out the square to the right.
- Mountain fold then unfold the two dashed lines labelled 1.
- Mountain fold along the four lines labelled 2.
- Valley fold along the four lines labelled 3.
- Bend your fortune teller along the 1 lines to make a plus shape, then open the flaps with SQUARE, TRIANGLE, etc written on them.
- Put your thumb and index finger on each hand into one of the holes under the flaps. You should be able to open the fortune teller in two ways: by pulling your hands apart, or by opening your fingers and thumbs.

Telling a fortune

- Get your friend to pick one of the four types of number written on the outside of the fortune teller.
- Open the fortune teller in the two ways alternately while spelling out the type of number.
- Get your friend to pick a number from four that are visible inside the fortune teller. The number they pick should be an example of the type they picked earlier.
- Open the fortune teller in the two ways alternately while counting up to the number they picked.
- Get your friend to pick a number of their type again.
- Open the flap inside the fortune teller. Your friend's fortune is under the number that they picked.

YOUR CORRESPONDENCE

20

We've got mail. Some of your emails, letters, and tweets.

Our fave

Excellent article as always from @chalkdustmag. Of course there are other options. I've always liked W^5 (meaning Which Was What Was Wanted).
And if QED doesn't feel nearly clever enough, try ancient Greek: ΟΕΔ.

— @sarahlovesmaths

Social media interaction: WWWWW.

Mathematically curious people are our kind of people.

— @Mamamakesbooks

Thanks for the praise! We're sure it's totally unconnected to our choice of winner for Book of the Year

---

Dear Chalkdust,

I like solitons very much. Please can you print my drawing of a soliton in your next issue so I can show all my friends at football club?

Thank you very much,
Christian (aged 32¼)

That's nice, Christian. We hope we haven't printed it upside down x

---

Get in touch!

Did you love one of our articles?
Found a new way to write down a triangle?
Spotted some ants solving a maths problem?
We'd love to hear about it!

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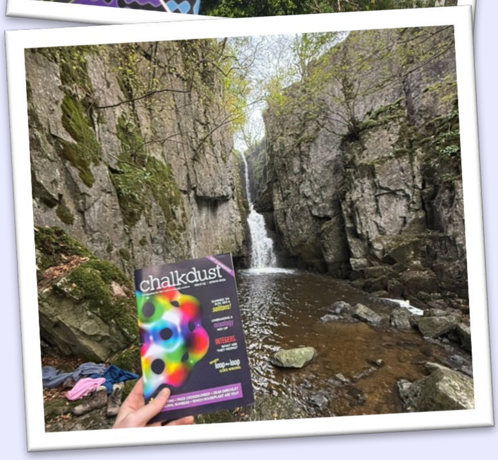
contact@chalkdustmagazine.com

CHALKDUST IN THE WILD



We love to see it! Send us your snaps and we'll print our favourites.

WE ASKED INSTAGRAM

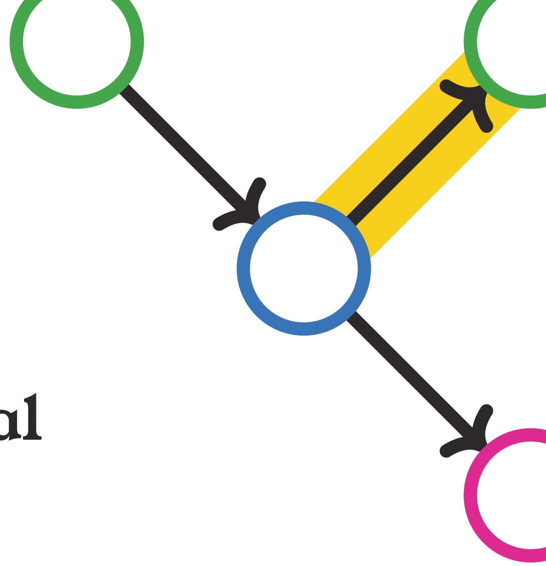
BLUE OR BLACK PEN?

YOUR ANSWERS

BLUE PEN: 24%

BLACK PEN: 76%

This one was surprisingly controversial at Chalkdust HQ!

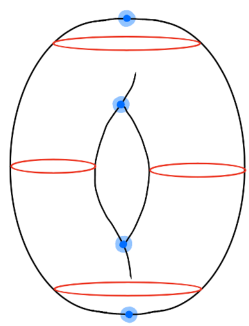


Treating the differential with discretion

Sophie Bleau explores the differential on Morseback.

THE differential is an interesting beast in many respects. It is a map which has been, ironically, integral to the theory of calculus, and has the fascinating property that its square disappears. This is to do with the fact that its partial derivatives always commute, so that they happen to cancel each other out. Last summer I had a go at proving why this happens using a combinatorial version of an existing proof by Michèle Audin, Mihai Damian and Reinie Erné. The goal was to use the subtle force that is Morse theory (sadly no relation to the code of dots and dashes).

Morse theory is all about analysing the topology on a manifold by looking at how differentiable functions act on it. For instance, I can look at the height function on a torus, and see that we have a maximum and a minimum as well as two saddle points. We can see that the type of level sets of the height function changes at each critical point. By level set we just mean a slice where the height function is constant. Any function with this property is called a Morse function, and the rigidity of this constraint means that we get stronger statements about them. For instance, if a Morse function yields exactly two critical points on a manifold then the manifold is homeomorphic to a sphere.



↑ A torus with the level sets of the height function marked, near the bottom of the torus the level sets are a circle, then between the saddle points the level sets are two circles, then finally between the upper saddle point and maximum, the level sets are a circle.

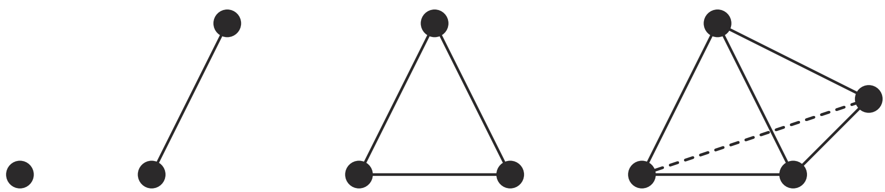
In the title I said we would treat the differential with discretion, by which I mean we translate our manifolds into the discrete setting. To do this, we can make the manifolds be simplicial complexes. If you don't know what a simplicial complex is, the name can look a bit intimidating but it is in reality just a graph, where each 'vertex' of the graph is a simplex. A simplex is just a fancy name for a triangle or one of its relatives in different dimensions. An n-simplex has $n+1$ vertices all connected to each other, so that a 0-simplex is a vertex, a 1-simplex is an edge, a 2-simplex is a triangle and a 3-simplex is a tetrahedron, and so on.

Doubling derivatives: it's trivial, darling

You may now be staring at your *Chalkdust* magazine and asking aloud "What do you mean the square of the derivative vanishes?" We are all familiar with computing the second derivative of a function—after all, how else would we analyse the nature of critical points. So, obviously, we cannot mean that taking the derivative twice always vanishes. Here when we say the square of the derivative we are referring to expressions more akin to those in vector calculus where we have identities like $\nabla \cdot (\nabla \times \boldsymbol{A}) = 0$ and $\nabla \times (\nabla f) = \mathbf{0}$. That is, the divergence of a curl vanishes and the curl of a gradient vanishes. As anyone who has gone through the *delight* of a vector calculus course knows, both of these identities rely on the fact that partial derivatives commute: for $f(x, y)$,

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right).$$

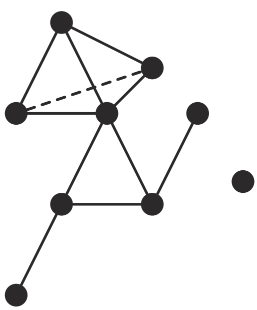
On the other hand, any differential geometers or algebraic topologists reading this will be saying "Of course taking the derivative twice vanishes, that is practically part of its definition."



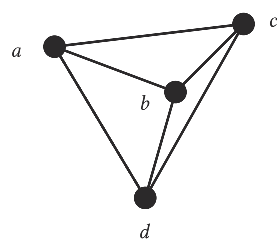
↑ From left to right: a 0-simplex, 1-simplex, 2-simplex, and 3-simplex.

Simple simplices

A simplicial complex is a gluing together of some simplices. For instance, in the image we have a 3-simplex glued to a 2-simplex, which is glued to two 1-simplices, and a disconnected zero simplex.

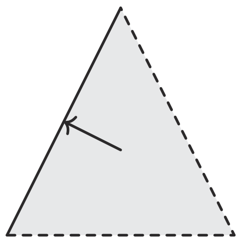


If you're wondering how a simplicial complex can represent a manifold, I can give you an example of the correspondence. A tetrahedron is the simplicial complex version of the sphere, where the discrete Morse function on the tetrahedron corresponds to the height function on the sphere, so that the 'critical maximum' and 'minimum' as we usually know them are represented by the 2-simplex at the top and the 0-simplex at the bottom. This is an example we will come back to in the next section.



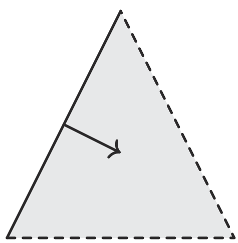
So what do functions on these objects look like?

In general, we want it to feel natural to slide down dimension, for instance, from a triangle to one of the edges on its boundary, as in the triangle below. Here I'm saying boundary to mean 'at the edge of the simplex'.



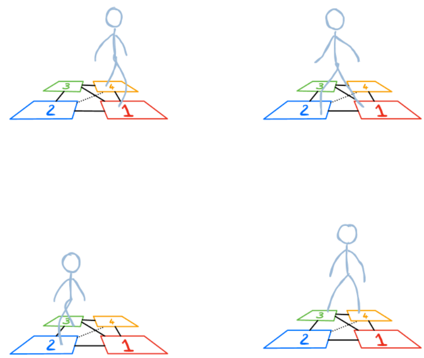
↑ We slide down a dimension from a 2-simplex to an edge on its boundary.

However, despite gravity, occasionally we might wish to jump up a dimension, for instance, from a boundary edge of a triangle to the triangle itself.

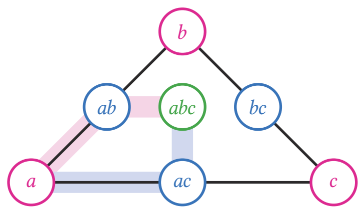


↑ We jump up a dimension from a 1-simplex to a triangle it lives on the boundary of.

We talk about orientations of simplices in everyday Morse discussions, which means defining an ordering on the vertices. On 1-simplices this is very intuitive as we are simply following the flow of the edge, like an arrow. On 2-simplices we take a circular route around the boundary of our triangle, and on 3-simplices we do a sort of 'box-step' dance between the vertices. Conventionally we label vertices by letters, a, b, c; edges by pairs of vertices, ab, bc, ca; and so on.



The Morse differential, as we will see, is the count of the signs of all the paths from an $(n+1)$-simplex to an n-simplex. In particular, if we take a 2-simplex α and drop down two dimensions to a vertex γ on its boundary, there are two paths to do this, shown in blue and in pink.



↑ In this diagram the 2-simplex is $\alpha = abc$, while the vertex γ is a.

As the direction of the blue path is the opposite of that of the pink, they have opposite signs, so that they cancel each other out. That is, the square of the Morse differential, found by dropping dimension twice, is 0.

A Morse function takes an n-simplex either up a dimension or down a dimension, and the rare instances of being able to jump up dimension we call Morse arrows. The distinguishing rule is that there can only ever be at most one Morse arrow associated to any simplex. If a simplex doesn't have any Morse arrows belonging to it, we call it a critical simplex. I like to think that it is critical of all the other simplices jumping about and being rowdy.

Hasse diagrams

A Hasse diagram is a directed graph whose vertices represent the simplices in a simplicial complex and whose directed edges represent boundary maps from an $(n+1)$-simplex to an n-simplex in the simplicial complex.

For our triangle abc the vertices are the 2-simplex abc, the three 1-simplices ab, bc, ca, and the three 0-simplices a, b, c. The directed edges show that ab, bc, ca are parts of the boundary of abc.

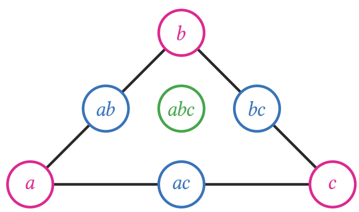
A nice way of displaying the information of possible directed paths is a Hasse diagram.

Below, we have (top left) a simplicial complex consisting of a 2-simplex abc and the simplices in its boundary. On the top right, we show the Hasse diagram, whose arrows correspond to boundary maps.

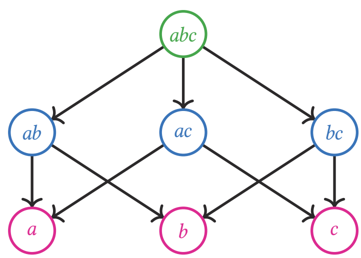
The Hasse diagram has only critical simplices, since none of the arrows point up. We can choose to make this slightly less trivial by changing the Morse function. This means reversing some of the arrows. Below, we see what effect this has on the simplicial complex, and on the right we see the modified Hasse diagram.

However, this is far too easy so let us glue two of these 2-simplices together and take an arbitrary Morse function on the resulting complex. This gives us the simplicial complex (bottom left) with modified Hasse diagram (bottom right) as shown.

Notice that this satisfies the rule that any individual simplex can only have one Morse arrow (highlighted in yellow) attached. Can you find the critical simplices in this Hasse diagram?

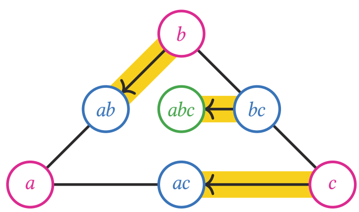


↑ A simplicial complex

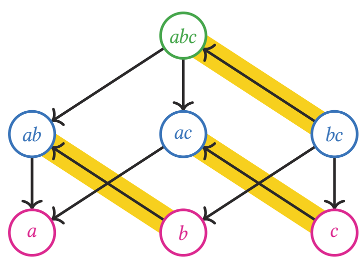


↑ The Hasse diagram

↓ The simplicial complex with a nontrivial Morse function



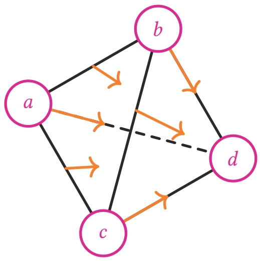
↓ The Hasse diagram with a nontrivial Morse function



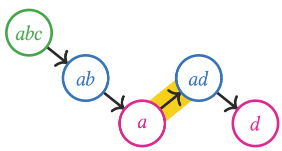
Go with the flow(line)

A directed path through a Hasse diagram between two critical simplices two dimensions apart is what we call a flowline, and forms the pivotal concept of this article. To see a flowline, let's look at a tetrahedron.

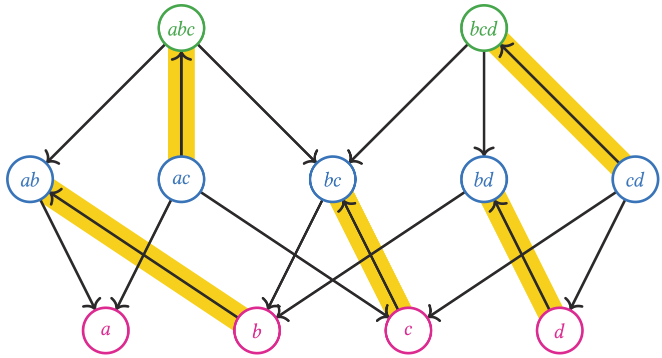
We alluded to this earlier: a possible Morse function we can define on the tetrahedron is that representing the height function on a sphere; imagine something like this:



If the top 2-simplex *abc* is a critical maximum (that is, a source simplex whose arrows all point away from it), and the 0-simplex *d* is a critical minimum (that is, a target simplex whose arrows all point toward it), the tetrahedron has the Hasse diagram to the right, and here's an example of a flowline:

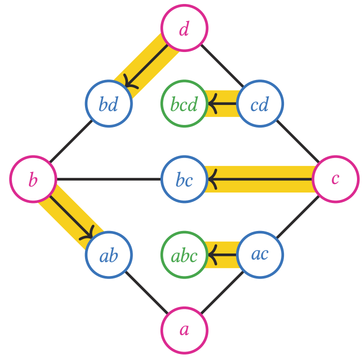


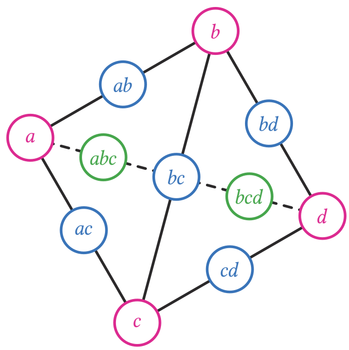
You see, we can define an algorithm which changes a flowline bit by bit in a small way to become another flowline, seeking out a flowline with a particular property. You may



↑ The Hasse diagram for…

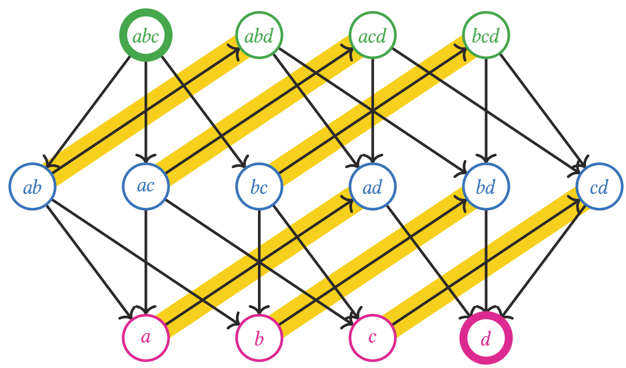
↓ …a possible Morse function on two glued triangles





↑ A tetrahedron…

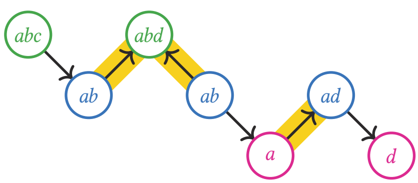
↓ …and the Hasse diagram where *abc* is a critical maximum and *d* is a critical minimum



notice that a flowline travels down and up a dimension almost in alternation, but to get to the level below it must drop down twice through an 'intermediate simplex'. The intermediate simplex in this flowline, for instance, is the 1-simplex ab. We want flowlines whose intermediate simplices are critical. If the flowline has this property, we say it is a critical flowline.

Using this concept of flowline criticality, we define the square differential of a critical $(n+1)$-simplex α in Morse theory to be the signed count of critical flowlines to critical $(n-1)$-simplices. This counts all of the paths that have a critical $(n+1)$-, n- and $(n-1)$-simplex. If there are no critical n-simplices or $(n-1)$-simplices, then $\partial^2\alpha = 0$ since there are no critical flowlines. But usually this is not the case, so we have to do a bit more to definitively prove our case.

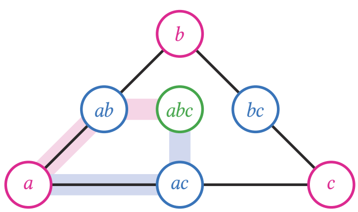
So what are the actions we can perform on a flowline? Actually, there are three. We can 'insert' a Morse arrow to the intermediate simplex, like so:

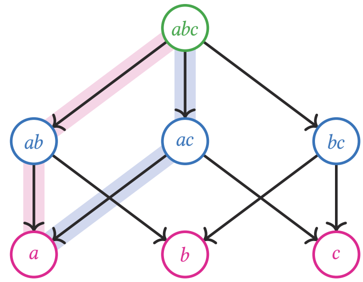


In general, insert adds in two Morse arrows either above the intermediate simplex, or below the intermediate simplex, depending on where its Morse arrow is (if it has one at all).

The second action we can perform on a flowline is a 'flop'. I showed you earlier that there are two ways down from the 2-simplex to a vertex on its boundary:

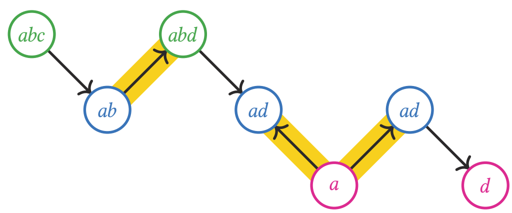
↓ The two ways down on a triangle





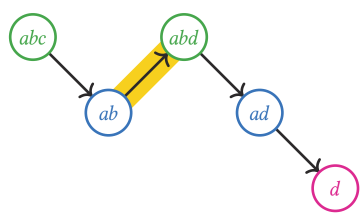
↑ The two ways down shown on a Hasse diagram

That is, if ab is our intermediate simplex, we can find the other intermediate simplex ac that is a boundary of abc and has a on its boundary. It's worth noting that if we flop the pink flowline once we obtain the blue flowline, and if we flop the flowline again we will get back to the pink flowline. We can generalise this. If β is our intermediate simplex, we can find the other intermediate simplex β' that is a boundary of α and has γ on its boundary (this β' is unique), and flop to this other path, like so:



In general, this flop action is always possible, and self-inverse, since applying it twice gets you back to the same flowline.

The final action is cancel. We can 'cancel' out the Morse arrow at an intermediate simplex by removing the redundant path adjacent to the intermediate simplex like so:

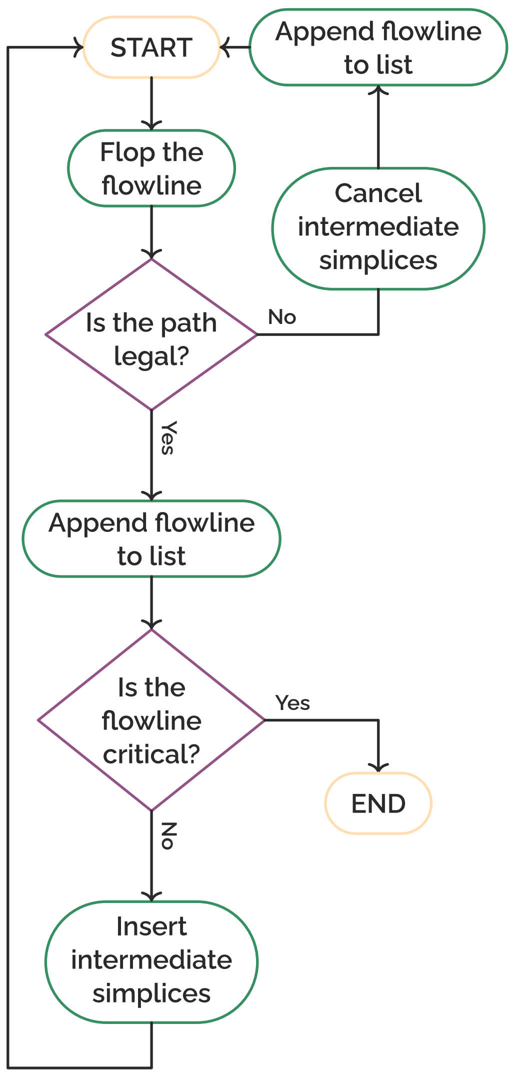


In general, we can cancel whenever there is a backwards Morse arrow.

Setting boundaries: know the signs

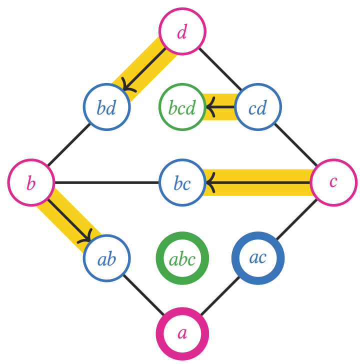
It's worth noting that cancel is inverse to insert, so if we devise an algorithm it doesn't make any sense to apply one after the other. Similarly, going back to our 2-simplex example, we can see that applying flop twice will get us back to the same flowline, so it doesn't make sense to ever apply two flops in a row.

Ultimately, the algorithm we find looks like this:



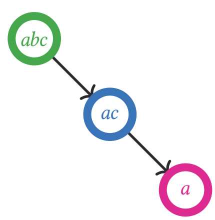
The algorithm in action

We sadly can't apply the algorithm to a critical flowline through the two glued triangles on page 65, because there aren't any critical flowlines. In particular, the differential as we define it in discrete Morse theory can only be applied to a critical simplex. In this simplicial complex, since our ingredients are 0-, 1-, and 2-simplices, and if we are dropping two dimensions, we require the existence of a critical 2-simplex in our simplicial complex. One thing we can do, however, is change the Morse function we endow it with in order to remove one of the Morse arrows.

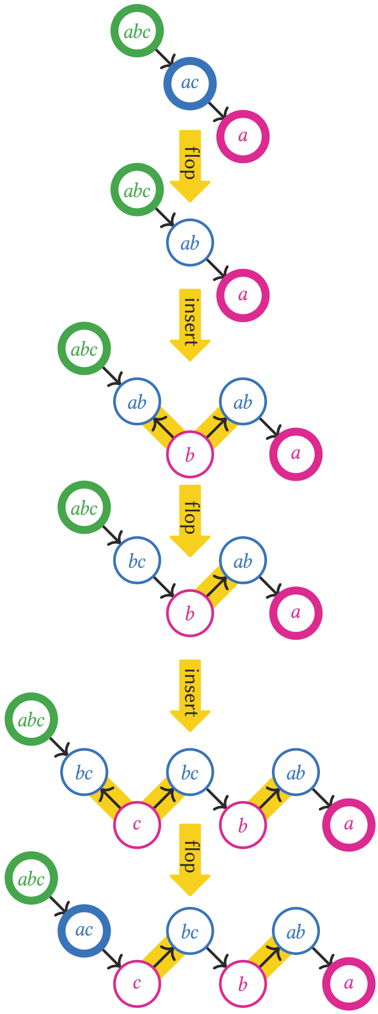


Now our simplicial complex has *abc*, *ac* and *a* as critical simplices. The astute reader may have noticed that I've highlighted critical simplices. This makes them easier to keep track of for the next step.

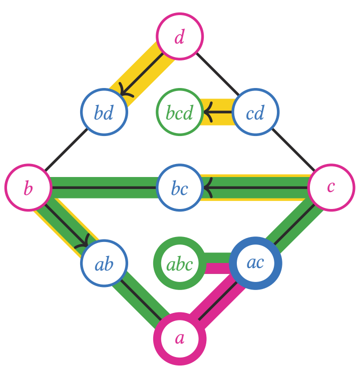
Let us start with the easiest critical flowline we can find:



Applying our algorithm to this critical flowline looks like this:



We can see from this that there are an odd number of floperations from start to finish, and that the critical flowlines travel through the simplicial complex in the following way:



The reason that the algorithm terminates at a critical flowline is because the only floperation we can apply to a critical flowline is flop, since there aren't any Morse arrows to insert or cancel.

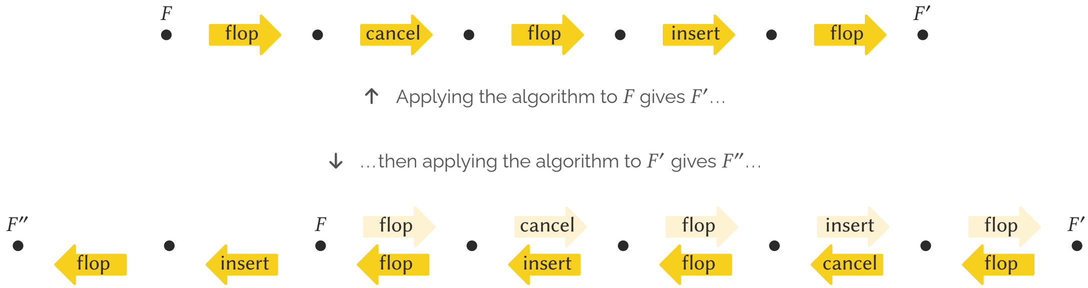
Notice that no matter how many times we cycle around each wing of the diagram, we'll always have an odd number of actions in our algorithm. This is going to be important later when we look at the signs of a flowline. But the punchline of this algorithm is that it terminates at a critical flowline. Not only that, but when we apply the algorithm to a critical flowline, the algorithm is involutive. This means that applying it once to some critical F' gives us a distinct critical flowline F'', and applying it twice gives us F' again. It is possible to find a flowline through the simplicial complex which isn't touched by this algorithm, but in this case when we apply the algorithm once and twice to this flowline, we will get a different pair of critical flowlines. In this way, we can partition all flowlines into equivalence classes.

Here, we note three details.

- The algorithm starts with a flowline F, and every flowline the algorithm passes through is 'equivalent' to F.
- The algorithm terminates when it arrives at a critical flowline.
- There are either 2 or 0 critical flowlines in any given equivalence class.

A consequence of this is that every critical flowline belongs to one equivalence class and there is a unique distinct flowline that is also in that equivalence class.

Another aspect of flowlines that we have only briefly touched on is the concept of a 'sign'. A flowline can be seen as positive or as negative, and each of the actions of flop, insert and cancel negates the sign of the flowline. We have talked a bit about why this happens for flop, but why do insert and cancel flip the sign of the flowline? Well, the sign of the flowline is calculated in terms of



(-1) to the power of half the number of arrows in the flowline. When we insert or cancel, we either increase or decrease the number of arrows by two, meaning that we change the power of (-1) by ± 1.

To see the space of flowlines, we could, for instance, have a flowline F, where applying the algorithm gives F', giving a sequence of flowlines and applying it twice gives F'', with a possible algorithm step-by-step shown above. In particular, we pass through the same flowlines to get from F'' to F' as we do to get from the F' to F''. You may be able to see, now, why the algorithm is involutive.

Now, if F has sign $+$ then we can find the sign of F' and F'' by looking at the number of actions between them. In particular, we can see that since there is always an odd number of actions (starting with flop and ending with flop) between F' and F'', they must have opposite signs.

Putting all this information together, we can find an expression for the squared differential in terms of flowlines through a simplicial complex. The squared differential of α, some $(n+1)$-simplex, is the signed sum of flowlines α to β times the number of flowlines β to γ for each critical simplex γ and each critical β.

For example, if we imagine a system with one critical γ and two critical flowlines F' and F'' through the system, then

$$\partial^2\alpha = \sum_{\substack{\text{critical} \\ \text{flowlines } F}} \operatorname{sign}(F) = \operatorname{sign}(F') + \operatorname{sign}(F'').$$

But since F' and F'' are the only two critical flowlines, they belong to the same equivalence class, and therefore are related by an odd number of operations so that $\operatorname{sign}(F') = -\operatorname{sign}(F'')$. Then $\partial^2\alpha = 0$.

In general, as critical flowlines come in these pairs, everything reduces to this case, so that $\partial^2\alpha = 0$ in any simplicial complex. This is a very satisfying way to see a proof of this geometric identity. In the smooth case, we have better contact with the manifolds themselves, but it's not so easy to actually compute things. When we see what surfaces look like in this combinatorial abstraction of the problem, we pinpoint the aspects of the space we really care about. And, as we have seen, by abstracting the problem to this discrete analogue, everything cancels out so nicely. So if you're ever feeling down, just remember that you can always abstract your problems. Discrete mathematics really does come through.

Sophie Bleau
Sophie is a PhD student at the University of Edinburgh. She spends her free time drawing Celtic knots and fangirling over Leonardo da Vinci. She has a cat called Norma, who takes great pleasure disrupting any mathematical progress—usually by sitting on it.

WIN A £100 GOODY BAG

The crossnumber

#20, set by **Humbug**

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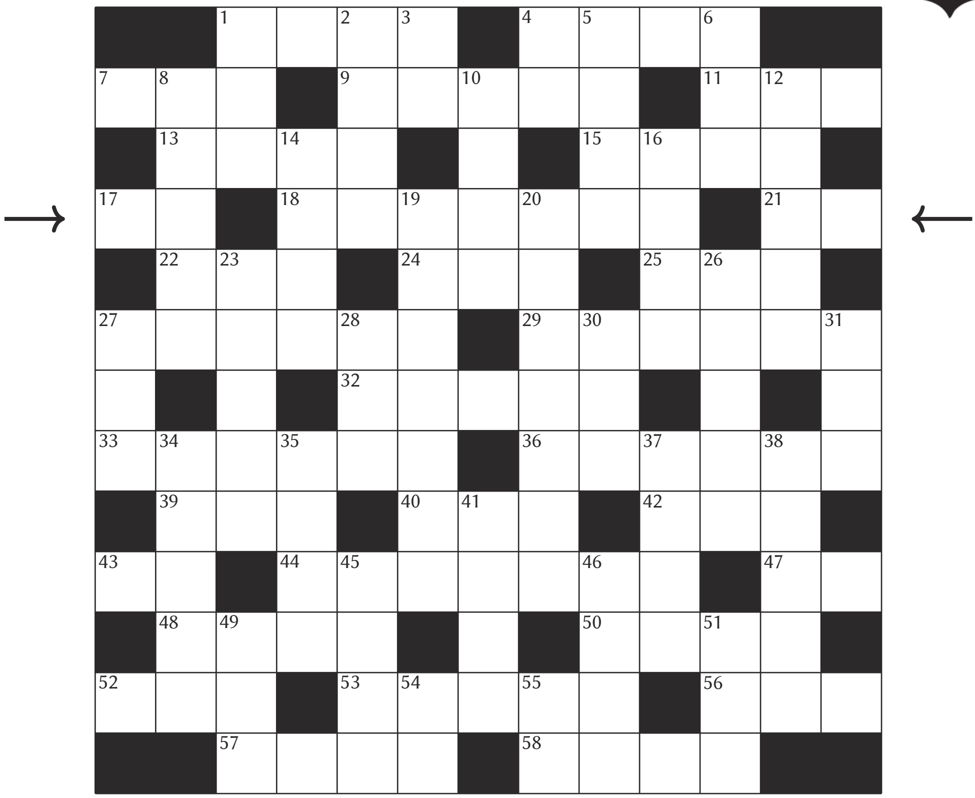


In the completed crossnumber:

- The number of entries that are even numbers is an even number;
- The number of entries that are odd numbers is an odd number;
- The number of entries that are prime numbers is a prime number;
- The number of entries that are square numbers is a square number;
- The number of entries that are multiples of 3 is a multiple of 3.

You may find the tick boxes by each clue helpful for keeping track of these requirements.

There is only one solution to the completed crossnumber. Solvers may wish to use the OEIS, Python, their ZX Spectrum, etc to (for example) obtain a list of prime numbers, but no programming should be necessary to solve the puzzle. As usual, no entries begin with 0.

To enter, send us the **sum of all the digits in the row marked by arrows** by **14 April 2025** via the form on our website (chalkdustmagazine.com). Only one entry per person will be accepted. Winners will be notified by email and announced on our blog by 1 June 2025. One randomly-selected correct answer will win a **£100 Maths Gear goody bag**, including non-transitive dice, a *Festival of the Spoken Nerd* DVD, a dodecaplex puzzle and much, much more. Three randomly-selected runners up will win a *Chalkdust* T-shirt.

Maths Gear is a website that sells nerdy things worldwide. Find out more at mathsgear.co.uk

Across

even	odd	prime	square	multiple of 3		Clue	
☐	☐	☐	☐	☐	**1**	A factor of 9999.	**(4)**
☐	☐	☐	☐	☐	**4**	A multiple of 32.	**(4)**
☐	☐	☐	☐	☐	**7**	22 more than 22A.	**(3)**
☐	☐	☐	☐	☐	**9**	Each digit of this number (except this first) is either equal to or one less than the previous digit.	**(5)**
☐	☐	☐	☐	☐	**11**	22 more than 42A.	**(3)**
☐	☐	☐	☐	☐	**13**	A multiple of 201.	**(4)**
☐	☐	☐	☐	☐	**15**	The sum of this number's digits is 13.	**(4)**
☐	☐	☐	☐	☐	**17**	Less than 33.	**(2)**
☐	☐	☐	☐	☐	**18**	Two times 44A.	**(7)**
☐	☐	☐	☐	☐	**21**	Six less than 47A.	**(2)**
☐	☐	☐	☐	☐	**22**	22 more than 56A.	**(3)**
☐	☐	☐	☐	☐	**24**	111 more than 40A.	**(3)**
☐	☐	☐	☐	☐	**25**	An anagram of two times 27D.	**(3)**
☐	☐	☐	☐	☐	**27**	Two times 29A.	**(6)**
☐	☐	☐	☐	☐	**29**	Two times 36A.	**(6)**
☐	☐	☐	☐	☐	**32**	34 more than a multiple of 100.	**(5)**
☐	☐	☐	☐	☐	**33**	Two times 27A.	**(6)**
☐	☐	☐	☐	☐	**36**	Two times 32A.	**(6)**
☐	☐	☐	☐	☐	**39**	A multiple of 9.	**(3)**
☐	☐	☐	☐	☐	**40**	A multiple of 111.	**(3)**
☐	☐	☐	☐	☐	**42**	22 more than 7A.	**(3)**
☐	☐	☐	☐	☐	**43**	30 less than a factor of 34D.	**(2)**
☐	☐	☐	☐	☐	**44**	Two times 33A.	**(7)**
☐	☐	☐	☐	☐	**47**	Six more than 21A.	**(2)**
☐	☐	☐	☐	☐	**48**	A multiple of 1.	**(4)**
☐	☐	☐	☐	☐	**50**	A multiple of 39.	**(4)**
☐	☐	☐	☐	☐	**52**	36 times 3D.	**(3)**
☐	☐	☐	☐	☐	**53**	Each digit of this number (except this first) is either equal to or one less than the previous digit.	**(5)**
☐	☐	☐	☐	☐	**56**	A multiple of 50.	**(3)**
☐	☐	☐	☐	☐	**57**	A multiple of 1000.	**(4)**
☐	☐	☐	☐	☐	**58**	One more than 57A.	**(4)**

Down

even	odd	prime	square	multiple of 3		Clue	
☐	☐	☐	☐	☐	**1**	The product of the digits of 28D.	**(3)**
☐	☐	☐	☐	☐	**2**	A multiple of 201.	**(4)**
☐	☐	☐	☐	☐	**3**	The sum of the digits of 28D.	**(2)**
☐	☐	☐	☐	☐	**4**	A square number.	**(2)**
☐	☐	☐	☐	☐	**5**	The sum of this number's digits is 12.	**(4)**
☐	☐	☐	☐	☐	**6**	22 more than 11A.	**(3)**
☐	☐	☐	☐	☐	**8**	A multiple of 203.	**(5)**
☐	☐	☐	☐	☐	**10**	A multiple of 9.	**(4)**
☐	☐	☐	☐	☐	**12**	A multiple of 9.	**(5)**
☐	☐	☐	☐	☐	**14**	Each digit of this number (except this first) is either equal to or one less than the previous digit.	**(4)**
☐	☐	☐	☐	☐	**16**	A multiple of 3.	**(4)**
☐	☐	☐	☐	☐	**19**	A multiple of 10.	**(7)**
☐	☐	☐	☐	☐	**20**	A multiple of 10.	**(7)**
☐	☐	☐	☐	☐	**23**	A multiple of 1001.	**(5)**
☐	☐	☐	☐	☐	**26**	A multiple of 9.	**(5)**
☐	☐	☐	☐	☐	**27**	An anagram of five times 25A.	**(3)**
☐	☐	☐	☐	☐	**28**	A multiple of 2.	**(3)**
☐	☐	☐	☐	☐	**30**	A multiple of 1.	**(3)**
☐	☐	☐	☐	☐	**31**	An anagram of 468.	**(3)**
☐	☐	☐	☐	☐	**34**	An anagram of 56789.	**(5)**
☐	☐	☐	☐	☐	**35**	A multiple of 5.	**(4)**
☐	☐	☐	☐	☐	**37**	A multiple of 5.	**(4)**
☐	☐	☐	☐	☐	**38**	A multiple of 601.	**(5)**
☐	☐	☐	☐	☐	**41**	A multiple of 1111.	**(4)**
☐	☐	☐	☐	☐	**45**	A multiple of 302.	**(4)**
☐	☐	☐	☐	☐	**46**	One of this number's digits is 1.	**(4)**
☐	☐	☐	☐	☐	**49**	Less than 333.	**(3)**
☐	☐	☐	☐	☐	**51**	The sum of this number's digits is 4.	**(3)**
☐	☐	☐	☐	☐	**54**	A multiple of 10.	**(2)**
☐	☐	☐	☐	☐	**55**	Three more than 54D.	**(2)**

TOP TEN

TOP TENS

10

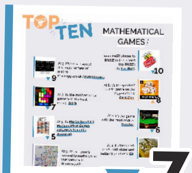
At number 10, it's the top ten for fans of pages 36–37: ***top ten maths-themed days out***.

9

At 9, it's great for when you're working out: ***top ten calculators***.

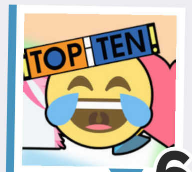
8

At 8, it's mathematician and journal editor ***top ten mathematical celebration days***.



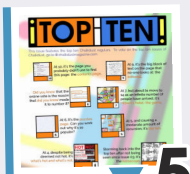
7

At 7, it's a winner: ***top ten mathematical games***.



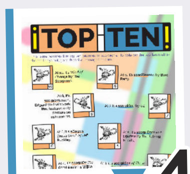
6

At 6, it's an online exclusive: ***top ten emoji to use in maths***.



5

At 5, it's one of two top tens on this list that features a top ten at number five: ***top ten Chalkdust regulars***.



4

Down 3 places to 4 as the joke's getting a little bit old: ***top ten pictures of scorpions***.

3

At 3, it's a retro hit: ***top ten haircuts***.

2

At 2, it's the essential reference tool for every lecturer: ***top ten colours of chalk***.


1

At the top of the charts, it's the top top ten and a new entry: ***top ten top tens***.

NEXT ISSUE

We rank the top ten **things to get in the post**. Vote now at chalkdustmagazine.com

ON MY BLACKBOARD...

We get to know the editors of *Chalkdust* by asking them about the most personal subject possible: the contents of their blackboards.

This issue, we talk to **Sophie Maclean**, who's been on the team since Issue 12.

What's the overall vibe?

'I really need to learn to say no to more things.'

Is there anything on there you'd never get rid of?

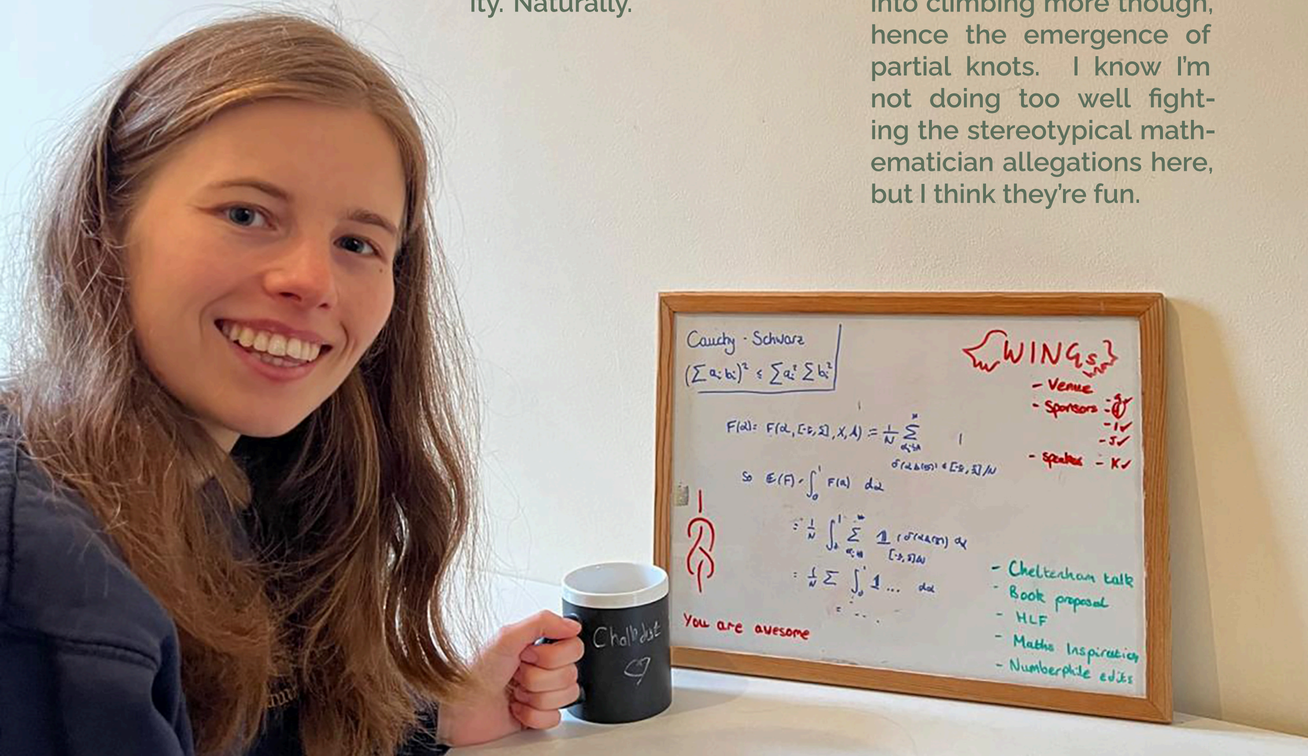
I'm hoping that some of my to-do list will vanish soon (although it seems to keep filling itself back up...).
I've always got some sort of positive message on there somewhere. And the Cauchy–Schwarz inequality. Naturally.

What's taking up the most space?

Definitely whatever I'm currently working on in my research! I'm doing a PhD in analytic number theory, so there's always some integral there.

Any drawings you're especially proud of?

I like to doodle lots of knots (in the mathematical sense). I've recently got into climbing more though, hence the emergence of partial knots. I know I'm not doing too well fighting the stereotypical mathematician allegations here, but I think they're fun.



Finally... chalkboard or whiteboard?

This is a tough one, but certainly at home a whiteboard, because chalkboards (and the floor below) are a nightmare to clean. But if you've ever done the thing where you draw a dotted line on a chalkboard, you'll understand why whiteboards will always be second best.

There is a good service operating on all other lines.

traditional

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